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KINEMATIC RELATIVITY

A SEQUEL TO
RELATIVITY, GRAVITATION AND WORLD STRUCTURE

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PREFACE

My earlier volume, *Relativity, Gravitation, and World-Structure*, was published in 1935. No sooner had it appeared than the subject underwent a rapid transformation, by which the merely kinematic results of the first volume were given a dynamical interpretation. In a series of eight papers published in the *Proceedings of the Royal Society* from 1936 to 1938, and in other papers published in the *Journal of the London Mathematical Society*, the *Astrophysical Journal*, the *Philosophical Magazine*, the *Proceedings of the Royal Society of Edinburgh*, the *Monthly Notices of the Royal Astronomical Society*, and especially in a joint paper with Dr. G. J. Whitrow published in the *Zeitschrift für Astrophysik* in 1938, I constructed on a kinematic basis a theory of dynamics, a theory of gravitation, and a theory of electromagnetism, isolated the two scales of time, and gave the work some philosophical interpretation. But no connected account of the general sequence of ideas in full detail has yet appeared, though an account of a substantial portion of the work was included in Dr. Martin C. Johnson's *Time, Knowledge, and the Nebulae* (Faber and Faber, 1945). The present volume aims at linking together the various scattered researches, but it is no mere summary of papers already published as it contains a reworking of the whole theory and many new results, in particular the expression of the t-equations of motion of a particle in Lagrangian form, and a novel application of the theory to photons which removes certain observational difficulties encountered by Hubble in his studies of the expanding universe.

The present volume is a sequel, not a substitute. It contains so much of the earlier volume as is necessary to make the present volume self-contained, but I have nothing of the earlier volume to withdraw. I do not consider that the many criticisms to which the earlier volume and the various later research papers have been subject deserve any detailed reply, as they rest mainly on either misunderstandings or prejudices, and it has been my object to avoid any note of controversy in the present volume as far as possible.

I should like to own my indebtedness to the work of Dr. G. J. Whitrow, whose helpful discussions of almost every point have been of great value. Dr. Whitrow has also contributed numerous original papers to the development of the subject. He has done me the final

kindness of reading through the proof-sheets of the present volume. An independent development of some of the lines of thought here presented, with fuller mathematical rigour, has been given by Professor A. G. Walker. Though I am not in agreement at all points with Professor Walker's line of development, I have benefited greatly by reading his papers.

E. A. M.

WADHAM COLLEGE, OXFORD
    June, 1947

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KINEMATICS

I

THE IDEAS OF KINEMATIC RELATIVITY

1. Origin of the investigations. Dynamics is the science of the description and theory of the motions of bodies. In order to describe the motion of a body, at least two concepts are necessary: the concept of a *frame of reference* and the concept of a *scale of time*.

A frame of reference is not a disembodied spirit. It must itself be defined with reference to the body or bodies whose motion it is proposed to discuss. The question at once arises whether there are any bodies in the universe which can be taken as fundamental frames of reference; are there any bodies whose motions are fundamental, in the sense that the simplest description of the motions of other bodies is obtained by using the first set as frames of reference?

The answer given by the orthodox theory of relativity is well known: it is that there are no fundamental frames of reference in the universe. *All* coordinate systems or frames of reference are equally valid for the description of the universe. The descriptions of individual examples of motion will be simpler or more complicated according to the frames of reference chosen; but the description of the laws of nature can be put in a form independent of the particular frame of reference chosen, and the same for all possible frames of reference. This answer, by its generality and by the *negative* character of its enunciation ('There are no preferential frames of reference')— negative statements (as pointed out by Sir Edmund Whittaker) have often proved of great value in the history of science, as in geometry and in thermodynamics—this answer, I say, has attracted the assent of some of the greatest minds of our time; and it has accumulated a considerable body of experimental evidence in its support. But it is impossible to establish a negative, by induction, with complete certainty. I should be the last to deny the abstract possibility of its truth. But an even more general answer is abstractly possible, and is worth consideration.

The contents, the material bodies, of the universe are not homogeneously distributed through space. Like city- and village-dwelling

mankind, the population of the universe is not spread uniformly, but is concentrated in aggregates known as *galaxies*, of which the great concourse of visible and invisible stars related to our own Milky Way forms one example. These galaxies seem to be the unit bricks out of which the universe is constructed. Many of them are spiral in structure, and are known as spiral nebulae. The more distant have not been resolved into stars by the telescope and camera, but the nearer ones have been shown, chiefly by the work of the great American observatories, to consist of stars, star-clusters, and gaseous nebular clouds, and to include Cepheid variable stars and novae, as these are observed in our own galaxy. Moreover, the different galaxies seem to be roughly comparable in size with one another. They are separated by vast distances, the intervening spaces being relatively devoid of matter. Each galaxy appears to have a centre, or nucleus. The various nuclei, judged by the displacement to the red of the lines in their spectra, appear to be all receding from us, and consequently from one another. The speed of recession increases with distance from ourselves, being, according to the researches of E. Hubble, directly proportional to the distance from ourselves. The average speed of recession is at the rate of 550 kilometres per second per million parsecs, a parsec being 3·26 light-years or $3·08 \times 10^{13}$ km., or $3·08 \times 10^{18}$ cm. The mass of each galaxy is of the order of a small multiple of 10^{11} suns. The linear dimensions of a typical galaxy, say, our own, are 15,000 to 20,000 light-years in radius, the shape being that of a flattened spheroid. The separations of the nuclei of the galaxies are of the order of 2 million light-years. The present mean density of the smoothed-out universe near ourselves is estimated as 10^{-28} gm. cm.$^{-3}$,[†] but, when allowance is made for obscuring matter and the increasing size of estimates of the masses of the nebulae, it may be as large as 10^{-27} gm. cm.$^{-3}$

These galaxies throng the fields of the most powerful telescopes, and long-exposure photographs show no apparent thinning out with distance. The faintest of them fade into an unresolved background. They are literally countless, and though some theories profess to give a figure for the total number of galaxies, my own belief is that the number is infinite. The reasons for this belief will appear in the sequel.

The galaxies constitute the natural frames of reference for the

[†] Hubble, *The Observational Approach to Cosmology*, 1937.

description of the motions of the contents of the universe. Their nuclei form a network of fundamental points, relative to any one of which the motions of any other bodies in the universe can be described. Whether we like it or not, we have no other natural frames of reference in the universe, and they are, in fact, being used by astronomers as the basis of the system of observed proper motions of the stars.

It therefore suggests itself that instead of assuming that *all* frames of reference are equivalent, we should make the less restrictive assumption that only the nuclei of the galaxies be taken as equivalent. We do not exclude the possibility that other frames of reference may be found to be also equivalent to these; we merely confine ourselves to deductions from the assumed equivalence of the galactic nuclei. That is to say, we can assume as a working hypothesis that the laws of nature, such as the law of motion of a free particle, are the same from whatever galactic nucleus they are described.

There is a further consideration, which militates against the view of orthodox relativity, that nature contains no preferential frames of reference. It has been the view of many thinkers, in particular Mach, that the laws of nature are a consequence of the contents of the universe being what they are; that the law of gravitation, for example, depends on the amount and distribution of the matter of the universe. Now the contents of the universe will have a different description according to the frame of reference used. The description of *the universe* by an observer at the centre of our own galaxy would be different, for example, from its description by an observer at the same place but moving with, say, one-third the speed of light. If so, we may expect that the descriptions of *the laws of nature* by two such observers would be different. They would be reconcilable, of course, but different. It is a consequence of this view that the laws of nature will only have the same descriptions when the universe has the same descriptions from the vantage-points of the observers in question. Only those vantage-points, frames of reference, observers—call them what you will—are equivalent from which the contents of the universe have the same description.

Whether the actual universe is such that its contents are described in the same way from every nebular nucleus taken as observing-point is very unlikely. But if so, then it is unlikely that the laws of nature have the same descriptions by such different observers. What we

need, to construct a science of laws of nature, is an ideal universe, in which the various nebular nuclei or *fundamental particles* provide identical descriptions of its contents. To the extent to which the actual universe approximates to the ideal one, to that extent will the laws of nature be describable in identical terms from the different nebular nuclei. To the extent to which the actual universe deviates from the ideal one, the laws of nature will be different as described from the different nebular nuclei; but we can readily proceed to a more realistic state of affairs, if we want to, by embroidering perturbations or variations on the ideal universe. It is necessary, however, first, to have a pattern, a norm of behaviour, a standard of comparison, before we can begin to discuss effects of non-homogeneity of description.

We take then, as fundamental points of reference, the nuclei of an idealized system of receding nebulae, such that the descriptions of the whole system, and consequently the laws of nature, are the same from all. This procedure, of starting with a universe in some sense *homogeneous*, has in fact been adopted by all cosmologists. It gives us a homogeneous stage for our play. It would be as purposeless and uninteresting to start with an irregular universe as it would be to enunciate geometrical theorems on the surface of an irregularly crumpled tablecloth. But it may be asked why we do not content ourselves with assuming that the laws of nature are the same from every nebular nucleus, with assuming in fact the ordinary form of the principle of uniformity of Nature, instead of assuming as well, in our idealized system, that the description of the contents of the universe is the same from each nebular nucleus. The answer is that this would preclude one of the main objects of our inquiry, which is to ascertain, in the broadest sense, the general laws of dynamics and related laws of nature. We do not wish to *assume* laws of nature, or take them for granted, or borrow them from experiment. We wish to infer, from the contents of the idealized universe, what would be the laws of nature in that universe. In order to talk about *laws* of nature at all, there must be a set of equivalent frames of reference; and if the laws of nature depend on the contents of the universe, then these contents, for the *idealized universe*, must have the same description for all fundamental observers.

It is to be noted that the kind of homogeneity we postulate for the idealized universe is a homogeneity of distribution of nebular nuclei, on the large scale. We do not ignore the fact that the matter

of the universe is concentrated in the vicinities of these nuclei, but we treat these nuclei as particles, and later discuss the smaller scale inhomogeneity occasioned by the actual distribution of matter near these nuclei.

It is further to be noted that the postulated homogeneity of distribution of nebular nuclei—the postulate that an observer at each such nucleus has the same view of the distribution of matter-in-motion in the universe—is not of the nature of an experimental or observational assumption at all. It is of the nature of a *definition*, a definition of the type of system it is proposed to consider. Just as we cannot expect to establish theorems in plane geometry without saying what we mean by an un-crumpled plane; just as we cannot expect to establish theorems in spherical geometry without introducing and defining a sphere, so we cannot expect to establish theorems in dynamics without first defining a system of frames of reference. We could choose different systems, but the system which first claims our attention is a system resembling the system of the galaxies, but defined to have an aspect of homogeneity. The interest of the resulting dynamics lies in the closeness of its resemblance to empirical dynamics, just as the interest of Euclidean geometry lies partly in its resemblance to empirical geometry. The mutually separating system of particles we define as constituting our frames of reference need not be pictured as *large* in the sense that the universe is large. We shall see that in some modes of its description it occupies the interior of an expanding sphere, and this need not be pictured as any bigger than a child's expanding balloon. No arguments based on its absolute size occur at any stage of our subsequent development.

2. Scale of time. The second desideratum in constructing a dynamics is a scale of time. But once we have determined on an expanding cloud of fundamental particles as our frames of reference, the possibility arises of using this same expanding cloud of particles to provide scales of time. *A priori* we can take any dynamical phenomenon whatever, and use its unfolding progress to define a scale of time. There is no natural *uniform* scale of time, because we cannot say what we mean by the word *uniform* in relation to time; we cannot catch the fleeting minute and put it alongside a later minute. Sometimes it is said that a uniform scale of time is defined by a periodic phenomenon. But this is to beg the question: it cannot tell us whether

two succeeding periods are 'equal'. But in a fundamental inquiry into the principles of dynamics it would be a culpable want of economy of thought to introduce a new dynamical phenomenon to give access to a scale of time. The system which has provided our frames of reference can be used to define a scale of time: if we can give a meaning to saying that the fundamental particles are all separating at *constant* velocities, this will give us a scale of time to work with. We shall have to examine whether, when two members *A* and *B* of the system are separating at constant velocity, then *A* and another member *C* can also be said to be separating at constant velocity. We must also examine whether we can attach a meaning to setting up the same scales of time at different places in the system. Though we cannot define *uniform time*, we shall find that we can define congruent time-keepers, at different places, by means of our system of mutually separating fundamental particles.

It may be asked at this stage whether we have not unduly circumscribed the minimum number of concepts necessary to construct a dynamics; is not a scale of *length* also necessary? The answer is that a method of *measuring* lengths is necessary, but that once a scale of time has been set up, the very fact that we need a method of perception of the objects whose distances we wish to ascertain compels us to consider *light*, and once the sending of light signals is at our disposal it is again uneconomic and redundant to introduce an independent scale of length. If we were to introduce a so-called rigid measuring-rod, we could not say what we meant by its maintaining the *same* length when transported, or when pointed in different directions. The ideally rigid measuring-rod is as incapable of definition as the clock measuring uniform time. To introduce further notions derived from experience, when the concepts of a cloud of receding particles together with the concept of light-signalling are sufficient, would be to depart from William of Occam's principle, *entia non sunt multiplicanda praeter necessitatem*. The most elementary analysis of the process of perception, combined with the individual's awareness of something he calls the *passage of time*, suffice, in conjunction with the separating cloud of particles, to provide measures of length and distance.

3. The substratum. The possibility of making progress in abstract dynamics with the aid of the concept of the expanding cloud of

fundamental particles arose from an accident of intuition. As explained in my earlier book, *Relativity, Gravitation, and World-Structure*, I was considering the observational phenomenon of the recession of the galaxies when I suddenly noticed that the explanation of this was obvious. A set of random motions, in a finite volume of otherwise empty space, will inevitably result in the expansion of the volume occupied by the moving particles. For the outward-moving particles will tend to cross the original boundary of the swarm, and the inward-moving particles will traverse the interior of the volume only to emerge at some other place. Moreover, if each member of the swarm is moving with uniform speed, when an adequate time has elapsed the fastest particles will be found on the outside of the now expanding swarm, followed by the next fastest, and so on, only the slower-moving members being in the vicinity of the original volume. There will be velocity-segregation, and the distances traversed by the particles from their original positions will be proportional to their velocities. The fact of the expansion, and the velocity-distance proportionality, are at once accounted for. Moreover, for such an expanding swarm, there is a natural zero of time, namely, the instant at which the system is first *given*, since, save in improbable circumstances of motion, the instant at which it is first given is also the instant of minimum volume of the system.

The next stage was one which would have occurred to anyone. It was to refine this picture of a swarm of particles in an isolated region in otherwise empty space, by imposing an aspect of homogeneity, of the type mentioned above. One simply assumed, as a matter of definition of the swarm to be considered, that the swarm did not contain any preferential particles. For the kinematics of the swarm one naturally assumed the kinematics of Einstein, embodied in the famous Lorentz formulae. The result was to obtain a velocity-distribution for the swarm, and a spatial-temporal distribution, which removed the objectionable feature of the original intuitive picture, namely, the being isolated in empty space, by making the swarm fill the whole of 'accessible' Euclidean space, the accessible portion being confined to the interior of an expanding sphere, which, by the properties of the Lorentz transformation, could be considered as having any member of the swarm for its centre.

The idealized system of mutually separating particles that resulted

I call now a 'substratum'. It has all the properties of infinite space,
in that a particle inside it, no matter how fast it moves, can never
reach its boundary. Its boundary is indeed entirely inaccessible
to its own members. The radius of this boundary, reckoned from
any arbitrary member of the system as centre, is proportional to the
time, as reckoned by that member at the instant under consideration.
The system is in a continual state of dilution with the flow of time,
due to its expansion; it is not homogeneous in density-distribution
in the view of any member of it, but it *is* homogeneous in the sense
that at the same epoch in the experience of any two particle-members
of it, the densities near them are the same. The substratum, being
a system of frames of reference in motion, plays the part for dynamics
that a plane plays for Euclidean geometry: it is the stage, the scene,
the theatre for the theorems of dynamics. Just as you cannot prove
theorems in geometry without being able to refer to a point any-
where in the Euclidean plane, so you cannot prove theorems in
dynamics without having at your disposal frames of reference every-
where in space. The substratum provides such frames.

4. Emergence of two scales of time. My former book, already
mentioned, was designed to explore the cosmological consequences
of the isolation of the substratum as a model of the expanding uni-
verse. But no sooner was it published than I found that I had hardly
begun to deal with the consequences for dynamics. It appeared
possible, as I have shown in numerous technical papers, to construct
accounts of dynamics, gravitation, and electrodynamics valid for the
substratum, and to relate these to Newtonian, Lagrangian, and
Hamiltonian dynamics, and Maxwellian electrodynamics. The most
important result that emerged was that the scale of time that is the
basis of Newtonian dynamics is not the scale of time in which the
universe is expanding, not, that is to say, the scale of time that is
the basis of the Lorentz formulae, or Maxwellian electrodynamics.
Einstein's dynamics, which uses the same scale of time for both
mechanics and optics, suffers in consequence from a confusion of
ideas which will be examined in the course of the present book.

When I was an undergraduate at Cambridge it was always said
by my fellow undergraduates that dynamics was a dead subject.
I hope that the investigations of this book will show how mistaken
we were in those days.

5. Contrast with classical physics. It must be recognized firmly by the reader that the ideas underlying the investigations in this book differ fundamentally from the ideas of mathematical physics as ordinarily understood; they make a clean breach with the ideas of traditional physics. The typical element in a branch of traditional physics is an empirical law of nature. Whether it is obtained, as Kepler's laws were obtained, by an inductive analysis of a mass of observational data, or whether it is obtained by a flash of inspired intuition, as Newton's laws of motion and Einstein's law of gravitation, it is essentially a statement of fact about the world, a statement from which consequences can be deduced, these further consequences having also the status of facts unless disproved by observation. Moreover, such laws of nature are usually enumerated in quantitative terms; usually, but not always. A branch of traditional theoretical physics contains the element of abstract reasoning, but its syllogisms are based on premises which are supposed to hold true in Nature.

But Kinematic Relativity, the name given to the class of ideas with which this book in part deals, does not begin with statements of quantitative fact. It does indeed assume, for each observer introduced, an awareness of something he calls the passage of time, by which he can place events in his own consciousness, that is, events constituting his own perceptions, in a temporal order. Without the incorporation into our work of this empirical but inescapable fact, there could be no description of change. A kinematics or dynamics would not be possible. Again, in this book I assume empirically that the number of spatial dimensions is *three*. It would be a simple matter to conduct the investigations of this book assuming any desired number of spatial dimensions; for example, by the methods of this book we could infer the form of the law of gravitation in a world of *n* spatial dimensions. But for reasons which will appear in a moment, I confine attention to the case of three spatial dimensions. Again, in order that the ego may discuss entities external to himself, it is necessary that he shall have means of perception. I call the means of perception *light*, but I assume no empirical properties of light. I also introduce the concepts of *particle* and *observer*. Such is the apparatus introduced. But this is far removed from assuming any empirical laws of nature. With this apparatus, I proceed to define systems of particles in motion, observed or capable

of being observed by observers, and then to derive theorems about them. The whole process is akin to the construction of an abstract geometry, only the elements in it, instead of being points and lines and surfaces, are particles in motion. The theorems are the undeniable consequences of the definitions (it is well known that axioms are concealed definitions). Just as the mathematician never needs to ask whether a constructed geometry is true, so there is no need to ask whether our kinematical and dynamical theorems are true. It is sufficient that the structure is self-consistent and free from contradiction; these are in fact the only criteria applied to a modern algebra or geometry. The interest then in the first instance lies in the revolutionary result that it actually proves possible to enunciate and prove theorems stating that in the presence of such-and-such abstract systems of particles such-and-such other particles will move in ways that can be specified. I say 'will move', not in any way appealing to empirical verification, but just meaning that such motions are logical consequences of the structure originally defined. No meaning can be attached to verifying a particular geometry, save in the sense of testing logically its self-consistency; and no meaning can be attached to *verifying* the dynamical theorems of this book. Many of them will appear very strange compared with the theorems of empirical dynamics, just as many of the theorems of non-Euclidean geometry seem strange and even absurd. Attempts were made in the early days of non-Euclidean geometry to pour ridicule on its results, and similar attempts have been made to pour ridicule on some of the kinematical and dynamical results exposed in this book. But theorems of non-Euclidean geometry are well known to lose their strangeness when it is realized that they are essentially theorems of Euclidean geometry stated for a non-flat plane. And the theorems of the dynamics based on Kinematic Relativity will be shown to lose their strangeness when the scale of time in terms of which they are stated is suitably transformed. Indeed, this is more than a parallel. The process by which theorems of Lobatchewskian geometry are translated into theorems of Euclidean geometry is essentially one by which the interior of a Euclidean circle is projected into the infinite Lobatchewskian plane, and the transformation of time-scale which we shall chiefly employ projects the interior of the initial expanding sphere of moving particles into the whole of an infinite space of hyperbolic character.

We superpose *motion* on *geometry*, and the result has in the first place all the abstract interest attaching to the construction of any geometry, with the enhancement of interest due to the incorporation of the novel feature of *motion*.

But in the second place, just as Euclidean geometry has a special interest from its happy coincidence with the empirical geometry based on particular empirical methods of measuring lengths, so the dynamics constructed in this book has a special interest from its wide field of agreement, when a suitable scale of time is chosen, with the empirical dynamics of tradition—the dynamics of Newton, Lagrange, Hamilton, and Einstein. This does not amount to the verification of the theorems; the theorems hold good in their own right. But it justifies the original choice of fundamental system of frames of reference, number of spatial dimensions, temporal sequence, and so on, used as concepts and definitions. Comparison with the empirical dynamics has in fact a much greater interest than comparison of Euclidean geometry with empirical geometry. For whilst nothing new emerges from the latter, a good deal that is new emerges from our comparison. The theorems of the dynamics of Kinematic Relativity do not always coincide in form with Einstein's refinements on empirical Newtonian dynamics, and suggest that in the long inductive journey from the observations that are the basis of orthodox relativity to the final equations of motion, there has been some going astray. This is not surprising when we consider how much deeper is the science of motion than the science of position, how much deeper is dynamics than geometry. We shall track down the discrepancies to a confusion between scales of time.

In the third place, the investigations have an interest in that they help to give an answer to the question, Why do the laws of dynamics hold good at all? Traditional physics, in starting with assumed laws of nature, debars itself at the outset from any possibility of answering this question. It can only be answered by starting with a set of definitions which lead to theorems closely corresponding to the laws empirically observed. There is then reason to believe that the definitions not only are self-consistent, but define abstract entities which are the counterparts of entities existing in Nature. We come back to the Platonic doctrine of ideas. The laws of dynamics hold good because particles in the presence of actual galaxies resemble

the abstract particles of our theory moving in the presence of abstract representations of galaxies. Only an abstract theory of motion can provide the nexus between the empirical facts and the why of the empirical facts.

To achieve so much, a considerable discipline is required from the physically minded reader. What is a commonplace to a mathematician is still unfamiliar to physicists. Physicists are not accustomed to reasoning about abstract ideas of things; the empirical is always intruding itself, openly or tacitly. To follow a kinematic argument one has to submit oneself to a process of self-denial: one must rigorously exclude all appeals based on intuition, explicit or implicit. Mathematicians brought up in the school of rigorous analysis, accustomed to proving, or more often disproving, the obvious, have found far less difficulty with the type of investigation contained in this book than the hard-headed experimental physicist, with his feet firmly planted on the ground. The intellectual climate in which an argument in Kinematic Relativity is conducted is markedly different from that in which an argument of current mathematical physics is conducted. In modern mathematical physics the investigator has results from the whole field of physics—mechanics, optics, thermodynamics, electrodynamics, quantum theory—at his disposal; he is not concerned with whether one result is or is not logically anterior to another. But in Kinematic Relativity we must not appeal to any result, law, or theorem not already established in the course of the investigation. No matter how repugnant to 'common sense', one must abide by the theorems obtained. For example, I was once asked how I reconciled such-and-such a result with the 'correspondence principle' of Bohr. But Kinematic Relativity is not acquainted with any principles. It is an entirely unprincipled subject. Until it progresses to the point of evolving its own form of the correspondence principle, this principle is irrelevant to its results. You might as well ask how a result of non-commutative algebra could be reconciled with the multiplication table.

This difference of climate accounts for the many irrelevant criticisms which have been so continuously showered on the investigations. I say 'irrelevant', for it is the exception for a criticism to be found helpful in correcting an actual mistake; and that is the only way in which a criticism can be legitimate. I could have avoided much of this criticism had I been content to proceed, at various

stages, as traditional physics proceeds. For example, I could have taken for granted the Lorentz formulae (as I did in my earliest investigations) as part of the established material of empirical physics. But to do so would have been to introduce an unnecessary empirical element from the outset; and, secondly, would have debarred me from identifying the time-variable occurring in them. Instead, I have throughout used them in the only sense in which I believe them to be valid, that is, in the contexts for which they have been established by Kinematic Relativity. Anyone who accepts them in a wider sense will be perfectly entitled to assent to the greater part of the investigations which follow. But he will not be in a position to turn each formula in the book back into immediate meaning in terms of sensory perception of light-signals.

6. Avoidance of concept of time-space. With these defensive remarks, I proceed to the formal investigations. It should be stated that nothing in this introductory chapter is to be taken *formally*. The words used to describe the essential ideas matter little; the ideas themselves shine out from the mathematics.

The variable chosen to measure the passage of time plays so fundamental a part in what follows that I have throughout distinguished between this variable and variables denoting position. That is, I have rarely used the concept of time-space. To speak of time as on a similar footing to the three space variables is almost always misleading, and it obscures the actual meanings of the various relations obtained. Our equations will often divide themselves into two groups, one a group expressing three-dimensional vector relations, the other scalar relations. It will prove essential to keep the two groups distinct. It will also appear that additional clarity is obtained by not sticking always to the restriction of expressing every relation in 4-vector form. Some of the most fruitful relations will be found to be three-dimensional ones which are not just the space-parts of 4-vectors.

I must make it clear, lastly, that I have nothing to retract from the investigations contained in my earlier volume already cited. The field of inquiry of the present volume, though based on the same set of ideas, is widely different. Emphasis is differently laid. Also, when the former volume was written I was completely ignorant of the directions the newer researches were to take.

Certain problems were left unsolved in the earlier volume, and the treatment there contained no attempt to translate the results into more traditional dynamical forms. But the spirit of the present volume is the same spirit that permeates the former volume, namely, the spirit of not assuming results not obtainable by the kinematic method.

TIME-KEEPING. THE LINEAR EQUIVALENCE

7. Temporal consciousness. Consider a single observer, an ego. He is conscious of something he calls the 'passage of time'. The phrase is to be taken as a unit. He is not conscious of something he calls 'time', and then aware of its passing. He means that if there are two events in his own consciousness, say, E_1 and E_2, then he can unfailingly say whether E_2 occurred 'after' E_1, or 'before' E_1, or 'simultaneously' with E_1. This is, of course, an over-simplification of the awareness, by the observer, of a temporal sequence. For E_2 may 'overlap' E_1, or be included in E_1. The simplification is analogous to the analysis of positions on a surface into 'points', or that of pieces of matter into 'particles'. Without a process of simplification or idealization or abstraction of this kind, initial progress would be unnecessarily complicated. We adopt, in fact, the undefined concept of a 'point-event' at the observer; only when we have got the theory resulting from the introduction of point-events in an advanced state would it be profitable to consider more complicated types of event. Let it suffice here to say that if actual events are deemed to possess a 'duration', we can define point-events as the beginning and ending of such durations, after the manner of Whitehead.

8. Definition of an arbitrary clock. We shall further conceive it possible for the observer to interpolate, between any two non-simultaneous events E_1 and E_2, occurring in his own consciousness, any number of further point-events; if E_2 is later than E_1, we can say that all the interpolated events are later than E_1 and earlier than E_2, and that they have the same 'before' and 'after' relations between one another as any actual pair of point-events have. We are thus led to the notion of a one-dimensional continuum of events at the observer. Choosing one of these events and labelling it zero, we can correlate all later events with the positive real numbers, and all earlier events with the negative real numbers, in such a way that the numbers t_1 and t_2 correlated with events E_1 and E_2 at the observer satisfy the relation $t_2 > t_1$ if E_2 is later than E_1. This correlation can be effected arbitrarily, subject to the condition just mentioned. Such an arbitrary correlation of events at the observer with real numbers we call a 'clock, arbitrarily graduated', and the number t

associated with any event E at the observer we call the 'epoch' of that event.

9. The first problem of time-keeping.

The first problem of time-keeping can then be stated as follows: If an observer or ego A has graduated his temporal consciousness in arbitrary fashion, i.e. has set up an arbitrary clock at himself, is it possible for a second observer B, in any kind of motion relative to A, to set up a clock which can be described as 'identical with', or, better, 'congruent to' A's clock, that is, to set up a clock which, in some sense to be made precise, can be said to keep the *same time* as A's clock?

For this problem to have a meaning it is necessary that A and B shall be able to inter-communicate; and for it to be capable of solution, it is necessary that A and B should be able to 'read' one another's clocks. We can picture A as assigning his temporal graduation of events at himself by setting up a 'clock-face' and arranging a 'hand' to run round it, in an arbitrary fashion to be arranged by himself. Observer B is to do the same at *himself*. Suppose now that B *sees* his own clock reading an epoch t_2' at the instant he *sees* A's clock reading an epoch t_1. Let B graph t_2' against t_1, obtaining a relation

$$t_2' = \theta(t_1). \tag{1}$$

Again, let observer A *see* his own clock reading an epoch t_4 at the instant he *sees* B's clock reading an epoch t_3'. Let A graph t_4 against t_3', obtaining a relation

$$t_4 = \phi(t_3'). \tag{2}$$

The most primitive case in which we can hope to set up clocks at A and B that may be called 'congruent' is when the relation of A to B is a symmetrical one. In that case we shall *define* the clocks as being *congruent* if the functions θ and ϕ are such that

$$\theta \equiv \phi. \tag{3}$$

(In the foregoing, the phrase 'at the instant he sees' means 'simultaneously with seeing', so that all that is required of either A or B is that he shall be able to make an immediate judgement of simultaneity between two perceptions, namely, perception of the clock at himself and perception of the clock that is not at himself.)

If relation (3) is not satisfied, the clocks as graduated are not congruent. The question arises, can B regraduate his clock so as to make it congruent with A's?

Let B regraduate his clock from reading t' to reading T', where

$$T' = \chi(t'), \qquad t' = \chi^{-1}(T').$$

The function χ must be a monotonic increasing function of its argument, to preserve the before-and-after relation for the newly graduated clock, and so possesses a unique inverse. Then

$$t_2' = \chi^{-1}(T_2'), \qquad t_3' = \chi^{-1}(T_3').$$

Hence relations (1) and (2) become

$$\chi^{-1}(T_2') = \theta(t_1), \qquad t_4 = \phi\chi^{-1}(T_3').$$

The first of these may be rewritten as

$$T_2' = \chi\theta(t_1).$$

Hence B's re-graduated clock will be congruent with A's if χ is such that

$$\chi\theta \equiv \phi\chi^{-1},$$

or

$$\chi\theta\chi \equiv \phi. \tag{4}$$

This is an operational equation for χ, given θ and ϕ. To solve it, put

$$\chi^* = \chi\theta.$$

Then

$$\chi^*\chi^* \equiv \phi\theta. \tag{5}$$

Hence the solution is obtained by taking the operational square root of the operational product $\phi\theta$. A method of effecting this operation has been given by Whitrow and the author.† It suffices here to state that a solution χ^* can be found, and that it can be arranged that this solution is everywhere differentiable. The solution holds good through any interval in which A and B do not meet. Once χ^* has been found, the regraduation function χ follows from

$$\chi = \chi^*\theta^{-1}.$$

10. Light signalling as acts of perception. We have shown how to set up identical or 'congruent' clocks at A and B solely from the observations that A and B can make on one another, without using the concept of the transmission of *light* between A and B. But it is to be noted that when A and B have set up congruent clocks, the clocks at any instant they are observed by either observer will not be giving the same readings. This is best explained by assuming that when A looks at B's clock, and notes its reading, he is making a perception of an event which occurred in some sense 'previous' to the event of the actual perception, and that the transmission of the knowledge of this event from B to A is effected by the propagation

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† *Zeits. für Astrophys.* **15**, 270, 1938.

of what is usually called *light*. If, in fact, darkness supervenes in the universe consisting of A and B, A will be unable to observe B's clock at all; therefore we may say that if A is enabled to read B's clock as reading t_3', then B has, at the instant t_3' by his own clock, sent a *light-signal* to A. Similarly, when B is enabled to read A's clock as reading t_1, then A has, at the instant t_1 by his clock, sent a light-signal to B. Again, we may suppose the instants t_2' and t_3' by B's clock to be simultaneous, i.e. $t_2' = t_3'$, in which case we may say that A has sent a light-signal to B at time t_1 by A's clock, that it has been reflected by B at the instant $t_2' = t_3'$ by B's clock, and returned to A so as to arrive at A at time t_4 by A's clock. This is in turn equivalent to saying that A strikes a light at time t_1 by his clock, and notes the reading t_2' ($= t_3'$) of B's clock when he sees it illuminated, and the reading t_4 of his own (A's) clock at the instant he sees B's clock illuminated. It will be noticed that the only physical property of light assumed in these procedures is that if B's clock is illuminated by A's action, then A sees it illuminated at a time by his (A's) clock which is not earlier than the time of the action of illuminating B's clock. This simple property can be itself deduced if we use a simple theory of causation, namely, that an effect cannot precede its cause. We also assume that if A performs an action which causes the illumination of B's clock, then he sees B's clock illuminated just once and only once.

It will be seen that to set up the relation between A and B of possessing congruent clocks, only the simplest acts of perception of A by B and B by A are required. Were these simple acts of perception excluded, A and B could scarcely be aware of one another's presence, and the problem of equipping them with congruent clocks could not arise. If we choose to be interested in the possibility of two different observers 'keeping the same time', we must permit them to announce to each other the times they are keeping, and our procedure involves nothing but this type of announcement. Thus the last person to quarrel with our suggested procedure should be the philosopher who reduces experience to the reception of sense-data.

11. The second problem of time-keeping. Suppose that we have an observer A, in possession of a clock arbitrarily graduated, and a number of other observers, B, C, D,... in any relative motion, and

that A, having observed B, C, D,..., communicates to them such information as enables them to graduate their clocks so that these are each separately congruent to A's clock. We write these relationships as

$$B \equiv A, \qquad C \equiv A, \qquad D \equiv A, \qquad ....$$

The relation of congruence being a symmetrical one we can equally well write

$$A \equiv B, \qquad A \equiv C, \qquad A \equiv D, \qquad ....$$

The second problem of time-keeping is then to find the conditions that $B \equiv C$, and so on for every pair. In words, we want the conditions that if B's clock is congruent to A's, and C's clock congruent to A's, then B's clock shall be congruent to C's, and so on. These conditions may be expected to take the form of restrictions on the relative motions of B, C, D,.... We solve this problem in stages.

Let A, B, C be three collinear observers who remain collinear. By this we mean that if B (supposed to be between A and C) at any epoch t_2' by his clock *receives* signals s_1 and s_2 from A and C respectively, then signals *leaving* B at epoch t_2' by B's clock *reach* C and A respectively at the same instants as the original signals s_1 and s_2 reach C and A respectively.

We suppose that $B \equiv A$ and $C \equiv A$. Let a signal which leaves A at t_1 by his (A's) clock reach B at t_2' by B's clock and reach C at t_3'' by C's clock; further, let the signal reflected by C at time t_3'' by C's clock reach B again at time t_4' by B's clock and reach A at t_5 by A's clock.

By the definition of clock congruence, since $B \equiv A$ there exists a function $\theta_{12}(t)$ such that

$$t_2' = \theta_{12}(t_1), \qquad t_5 = \theta_{12}(t_4').$$

Since $C \equiv A$, we have also

$$t_3'' = \theta_{13}(t_1), \qquad t_5 = \theta_{13}(t_3'').$$

We shall call the functions θ_{12}, θ_{13}, where the first suffix is smaller than the second, *signal functions of the first kind*; and we shall write their functional inverses as θ_{21}, θ_{31}, and call them *signal functions of the second kind*. A signal function of the first kind, it will be seen, gives the time of reception as a function of the time of emission, by the two observers concerned; a signal function of the second kind gives the time of emission as a function of the time of reception, by the same observers.

From the above relations,

$$t_3'' = \theta_{13}\theta_{21}(t_2'), \qquad t_4' = \theta_{21}\theta_{13}(t_3'').$$

The observers B, C will be *equivalent*, or possess congruent clocks, if t_3'' is the same function of t_2' as t_4' is of t_3'', i.e. if

$$\theta_{13}\theta_{21} \equiv \theta_{21}\theta_{13}. \tag{6}$$

When this condition is satisfied, we write

$$\theta_{13}\theta_{21} \equiv \theta_{21}\theta_{13} \equiv \theta_{23}. \tag{7}$$

The relation (6) imposes a restriction on the function θ_{13} given the function θ_{21} (or θ_{12}). Now the function θ_{13} is in some way a measure of the 'motion' of C relative to A. (When we come to introduce coordinates, we shall put this measure of 'motion' into analytical form.) Similarly, θ_{12} measures the motion of B relative to A. It follows that for C's clock to be congruent with B's, there must be a restriction on the motion of C relative to A. In words, if, when A has graduated his clock arbitrarily, and B, an observer in motion relative to A, has graduated his clock so as to be congruent with A's clock, a further observer C, collinear with A and B, graduates his clock so as to be congruent to A's clock, then C's and B's clocks, as thus graduated, will be congruent only if there is a certain restriction on C's motion, the restriction depending on the motion of B relative to A. This restriction is contained in the commutation relation (6).

It is now easily verified from identities (7) that of the three functions θ_{12}, θ_{13}, θ_{23} and their inverses, any one commutes with any other.

Consider now a set of observers A_p such that any three remain collinear and such that any pair of them can graduate their clocks so as to be congruent. We proceed to show that *any* pair of the corresponding signal functions commute with one another. For, since every pair possess congruent clocks, for any suffixes p, q, r

$$\theta_{pr} \equiv \theta_{pq}\theta_{qr} \equiv \theta_{qr}\theta_{pq}; \tag{8}$$

hence

$$\begin{aligned}
\theta_{pq}\theta_{rs} &\equiv (\theta_{ps}\theta_{sq})\theta_{rs} \\
&\equiv \theta_{ps}\theta_{rs}\theta_{sq} \\
&\equiv \theta_{ps}(\theta_{rp}\theta_{ps})(\theta_{sp}\theta_{pq}) \\
&\equiv \theta_{rs}\theta_{pq}. \tag{9}
\end{aligned}$$

12. Solution of the commutation identities.

It was shown by Whitrow[†] how to find the general solution of the identities (9). The form of these identities suggests that we seek solutions θ of the identity

$$\theta\theta_0 = \theta_0\theta \qquad (10)$$

(where θ_0 is a given function), which themselves commute in pairs. It is easily found that such solutions θ form a group. We shall assume that this group contains an infinitesimal member, i.e. one differing by as little as we please from the identical operator. Let the function obtained by operating with the infinitesimal member on t be

$$t + \epsilon\omega(t),$$

where ϵ is small. Take this to be a θ satisfying (10). Then

$$\theta_0(t) + \epsilon\omega\theta_0(t) = \theta_0\{(t + \epsilon\omega(t)\},$$

whence expanding the right-hand member by Taylor's theorem and letting $\epsilon \to 0$ we get

$$\frac{\theta_0'(t)}{\omega\theta_0(t)} = \frac{1}{\omega(t)}.$$

Define a new function $\Omega(t)$, of inverse $\Omega^{-1}(t)$, by

$$\Omega^{-1}(t) = \int \frac{dt}{\omega(t)}.$$

Then

$$\int^t \frac{dt}{\omega(t)} = \int^t \frac{\theta_0'(t)}{\omega\theta_0(t)}\,dt = \int^{\theta_0(t)} \frac{dT}{\omega(T)},$$

where

$$T = \theta_0(t).$$

Hence

$$\Omega^{-1}(t) + \text{const.} = \Omega^{-1}\theta_0(t).$$

Writing this as

$$\Omega^{-1}\theta_0(t) = \Omega^{-1}(t) + \lambda_0,$$

where λ_0 is any constant, we have, on operating on each side with Ω,

$$\theta_0(t) = \Omega\{\Omega^{-1}(t) + \lambda_0\},$$

or, putting $\Omega(t)$ for t,

$$\theta_0\Omega(t) = \Omega(t + \lambda_0).$$

Given θ_0 (monotonic increasing) and taking $\Omega(t)$ arbitrary in the interval $0 \leqslant t \leqslant \lambda_0$, subject to $\Omega(\lambda_0) = \theta_0\Omega(0)$, we can obtain the value of Ω for any value of t; the Ω so constructed will be monotonic increasing, and so will possess a unique inverse. Hence $\omega(t)$ can be

† *Quart. Journ. Math. (Oxford)*, 6, 249, 1935.

found. Any other θ must commute with the infinitesimal member, and so must be of the form found for θ_0, namely,

$$\theta(t) = \Omega\{\Omega^{-1}(t) + \lambda\}, \tag{11}$$

where λ is some new constant. It is readily verified that any two θ's of this form do in fact commute with one another. For example,

$$\begin{aligned}
\theta_1\,\theta_2(t) &= \Omega\{\Omega^{-1}\theta_2(t) + \lambda_1\} \\
&= \Omega\{\Omega^{-1}(t) + \lambda_2 + \lambda_1\} \\
&= \theta_2\,\theta_1(t).
\end{aligned}$$

Thus (11) is the complete solution of the identities (9) on the assumption made. If we put

$$\Omega(t) = \psi(e^t), \qquad \alpha = e^\lambda,$$

then

$$\Omega^{-1}(t) = \log \psi^{-1}(t),$$

and so

$$\begin{aligned}
\theta(t) &= \psi(e^{\lambda + \Omega^{-1}(t)}) \\
&= \psi\alpha\psi^{-1}(t). \tag{12}
\end{aligned}$$

13. Linear equivalence. We now define a *linear equivalence* as the set of collinear observers whose signal functions $\theta_{pq}(t)$ are given by

$$\theta_{pq}(t) = \psi\alpha_{pq}\psi^{-1}(t), \tag{13}$$

where ψ is any given (monotonic increasing) function characteristic of the whole equivalence and α_{pq} is a positive real number characteristic of the pair of observers corresponding to θ_{pq}, and α_{pq} takes all positive values.

It follows that since

$$\theta_{qp} = \psi\alpha_{qp}\psi^{-1},$$

and since $\theta_{qp} = \theta_{pq}^{-1}$, we have

$$\alpha_{qp} = 1/\alpha_{pq}.$$

Moreover,

$$\alpha_{pr} = \alpha_{pq}\alpha_{qr}.$$

Since Ω and so ψ are partly arbitrary, any number of linear equivalences can be constructed containing two given observers.

14. Equivalence defined by three collinear observers. We now prove that in general, if three observers belong to a linear equivalence, then this linear equivalence is unique.

Suppose that the signal function connecting the *two* observers A, B is expressible in the two forms

$$\Omega\{\Omega^{-1}(t) + \lambda\},$$

and

$$\Omega^*\{\Omega^{*-1}(t) + \mu\}.$$

This means that

$$\Omega\{\Omega^{-1}(t)+\lambda\} = \Omega^*\{\Omega^{*-1}(t)+\mu\}$$

for all t. For t put $\Omega(t)$ and then perform the operation Ω^{*-1} on each side. Then

$$\Omega^{*-1}\Omega(t+\lambda) = \Omega^{*-1}\Omega(t)+\mu.$$

Put

$$\Omega^{*-1}\Omega \equiv \Phi,$$

or

$$\Omega = \Omega^*\Phi.$$

Then

$$\Phi(t+\lambda) = \Phi(t)+\mu.$$

The general solution of this is

$$\Phi(t) = at+\Theta(t),$$

where

$$a\lambda = \mu,$$

and Θ is a periodic function of period λ, so that

$$\Theta(t+\lambda) = \Theta(t).$$

Now C is a member of a linear equivalence containing A and B. Hence the signal function connecting A and C can be written in the two forms

$$\Omega\{\Omega^{-1}(t)+\lambda'\}, \qquad \Omega^*\{\Omega^{*-1}(t)+\mu'\}.$$

Accordingly, by the same argument as before,

$$a\lambda' = \mu', \qquad \Theta(t+\lambda') = \Theta(t).$$

Hence if λ and λ' are mutually incommensurable, $\Theta(t)$, having two incommensurable real periods, must reduce to a constant. Hence $\Phi(t)$ is of the form

$$\Phi(t) = at+b.$$

Hence

$$\Omega(t) = \Omega^*(\Phi) = \Omega^*(at+b).$$

Hence

$$\Omega^*(t) = \Omega\{(t-b)/a\},$$

whence

$$t = \Omega\left(\frac{\Omega^{*-1}(t)-b}{a}\right),$$

or

$$\Omega^{*-1}(t) = a\Omega^{-1}(t)+b.$$

Hence the set of signal functions

$$\Omega^*\{\Omega^{*-1}(t)+\Lambda\}$$

reduce to

$$\Omega^*\{a\Omega^{-1}(t)+b+\Lambda\},$$

i.e. to

$$\Omega\left(\frac{a\Omega^{-1}(t)+\Lambda}{a}\right),$$

i.e. to

$$\Omega\{\Omega^{-1}(t)+\Lambda/a\}.$$

But this is the set of signal functions generated by Ω.

15. A linear equivalence is a kinematic entity, a definite corpus of relationships, and it plays a fundamental part in time-determinations. Just as geometry involves definitions of points, lines, and planes (possibly through being the subjects of axioms) which play a part in the subsequent theorems, so time-keeping involves the introduction of linear equivalences. Physically, a linear equivalence is a collinear set of observers who can be equipped with compatible clocks. We shall give examples of specific linear equivalences later.

16. **Main theorem.** We shall now prove that given two linear equivalences generated by the functions ϕ and ψ, the ϕ-equivalence becomes identical with the ψ-equivalence on regraduating the clocks of the members of the ϕ-equivalence in an appropriate way.

For let ϑ_{pq} be a typical signal function of the ϕ-equivalence. Then by the definition of the ϕ-equivalence,

$$\vartheta_{pq}(T) = \phi\alpha_{pq}\phi^{-1}(T),$$

where T is the time kept by a member of the ϕ-equivalence. Now regraduate the clocks of the members of the ϕ-equivalence so that any clock-reading T is renumbered t, where

$$t = \chi(T),$$

χ being a monotonic function possessing a unique inverse. The definition of a signal function implies that if a signal leaves P at clock-reading T_1 by P's clock, is reflected by Q at clock-reading T_2' by Q's clock, and returns to P at clock-reading T_3 by P's clock, then

$$T_2' = \vartheta_{pq}(T_1) = \phi\alpha_{pq}\phi^{-1}(T_1),$$
$$T_3 = \vartheta_{pq}(T_2') = \phi\alpha_{pq}\phi^{-1}(T_2').$$

If t_1, t_2', t_3 denote the epochs of the same events when the clocks have been regraduated, we have

$$t_2' = \chi\phi\alpha_{pq}\phi^{-1}\chi^{-1}(t_1),$$
$$t_3 = \chi\phi\alpha_{pq}\phi^{-1}\chi^{-1}(t_2').$$

But these relations represent a linear equivalence amongst the clocks reading t with signal functions θ_{pq} given by

$$\theta_{pq} = \chi\phi\alpha_{pq}\phi^{-1}\chi^{-1}.$$

This equivalence will be identical with the given ψ-equivalence if

$$\chi\phi\alpha_{pq}\phi^{-1}\chi^{-1} \equiv \psi\beta_{pq}\psi^{-1}$$

for some correspondence between the α's and β's. A sufficiently general solution of this identity is

$$\chi(t) = \psi[k\{\phi^{-1}(t)\}^s] \tag{14}$$

with

$$\alpha_{pq}^s = \beta_{pq}, \tag{15}$$

k and s being arbitrary.

Thus a linear equivalence remains a linear equivalence on clock regraduation. And essentially there is *only one* linear equivalence. All apparently different linear equivalences, generated by different functions ψ, are merely different descriptions of the same kinematic entity.

17. Coincidence at a point. It will now be shown that if two members of an equivalence coincide at an epoch t_1, then all the members coincide at this epoch. For if the observers P and Q coincide at epoch t_1 by their clocks, then, by the definition of signal functions, at this epoch

$$\theta_{pq}\theta_{pq}(t_1) = t_1.$$

Hence if ψ is the generating function of the equivalence,

$$\psi\alpha_{pq}^2\psi^{-1}(t_1) = t_1.$$

Hence

$$\alpha_{pq}^2\psi^{-1}(t_1) = \psi^{-1}(t_1).$$

Now $\alpha_{pq}^2 \neq 1$, for if α_{pq} were unity the two members of the equivalence would be identical at all epochs. Hence

$$\psi^{-1}(t_1) = 0,$$

or

$$\psi(0) = t_1.$$

Now consider any other two observers, P', Q', members of the equivalence, with parameter $\alpha_{p'q'}$. Then

$$\theta_{p'q'}\theta_{p'q'}(t_1) = \psi\alpha_{p'q'}^2\psi^{-1}(t_1) = \psi(0) = t_1,$$

and thus P' and Q' coincide at epoch t_1.

It is not a necessary property of an equivalence that its members possess an epoch of common coincidence; but if two observers ever coincide, all coincide at that epoch.

18. Introduction of coordinates. To translate the definition of an equivalence into our usual ways of describing *motion*, it is necessary to introduce conventions by which an observer A can assign *coordinates* to an observer B from his observations of B. This we now investigate.

If an observer A sends a light-signal at epoch t_1 by his clock, and sees B at time t_3 by the same (A's) clock (i.e. receives the reflected signal at time t_3), then in some way the difference between t_3 and t_1 is a measure of the separation of B from A. For the 'farther' B is from A, the longer will it be before the return signal is received by A. Moreover, the average of t_3 and t_1 affords some measure of the epoch A will be disposed to assign to the event of reflection at B. Observer A can, of course, construct two independent numbers out of his observations t_1 and t_3 in an infinite variety of ways. But their difference and sum have each a property which makes them respectively appropriate as measures of 'distance' and 'epoch' of an event not at A. For a constant added to the zero of A's clock leaves the difference of the observations unaltered, and it adds the same constant to the average of the observations. Thus the measure of distance would be unaffected by the change of zero of the clock used by A, and the measure of epoch would be increased by the same constant.

Now let A choose a positive number c. Let him define as the *epoch* of the event of reflection at B of the light-signal the number t given by
$$t = \tfrac{1}{2}(t_3 + t_1), \tag{16}$$
and let him define as the *distance* of the same event from himself the number r given by
$$r = \tfrac{1}{2}c(t_3 - t_1). \tag{17}$$

The numbers t and r are called *coordinates* of the event of reflection as reckoned by A using his own clock. It is clear that such coordinates are conventional constructs. But such conventional constructs can always be immediately transformed back again into the observations out of which they arose by the formulae
$$t_1 = t - r/c, \qquad t_3 = t + r/c. \tag{18}$$

Until *distance* and *epoch* have been defined, it is impossible to define velocity. But we can now define the velocity of a particle (in the line of sight) as dr/dt, where r and t have the above meanings, and r is considered as a function of t.

19. Velocity of light. Consider the set of associated values of r and t corresponding to a signal sent out by A at the *fixed* instant t_1. The rate of increase of r with respect to t for this signal will be defined as the velocity of light for this signal. The value of dr/dt for fixed t_1

measures in fact the rate of increase of the *distance* to which the signal has been propagated by the epoch t, according to A's clock. By (18), for this set of values of r and t,

$$r = c(t-t_1), \qquad \frac{dr}{dt} = c. \tag{19}$$

This is the signal velocity assigned by A.

20. Observer B can perform similar observations with regard to A, and assign coordinates (r', t') to events at A. By agreement he chooses the same positive number c for converting clock-differences into distance coordinates. It follows that, with these conventions, B will also assign to the signal velocity the value c.

21. Epoch-distance relation and clock-running relation. Let us consider in more detail the observations which A can make on B. Let A make a set of observations of r and corresponding values of t for some actual motion of B relative to A, and let him plot r against t, obtaining a function, say, $r = c\phi_{12}(t)$. This will be called the *epoch-distance relation* for B's motion as observed by A.

Further, let A observe the reading of B's clock at the event of the reception by B of A's signal. This will be the actual reading of B's clock to A at the moment of illumination of B's clock. Let t' be the reading of B's clock at the event to which A assigns the epoch-coordinate t. Let A plot t' against t, obtaining a relation, say, $t' = f_{12}(t)$. This will be called the *clock-running relation* for B's clock as compared with A's, in A's experiences.

Let B record his observations of A similarly. Let $c\phi_{21}(t)$ and $f_{21}(t)$ be the *epoch-distance* relation and the *clock-running* relation for A's motion relative to B, as observed by B. Suffixing epochs and distances by the observer at which events occur, and using primes to distinguish B's assignments or observations, we have

$$r_B = c\phi_{12}(t_B), \qquad t'_B = f_{12}(t_B),$$
$$r'_A = c\phi_{21}(t'_A), \qquad t_A = f_{21}(t'_A).$$

Here t'_B and t_A are observed clock readings; r_B, t_B are A's assignments of coordinates to events at B; r'_A, t'_A are B's assignments of coordinates to events at A.

If now the relation of B to A is a symmetrical one, and if B's

clock has been graduated so as to be congruent to A's, then we must have

$$\phi_{12} \equiv \phi_{21}, \qquad f_{12} \equiv f_{21}.$$

Call the first ϕ, the second f.

Now consider a light-signal which is dispatched by A at time t_1 by A's clock, reaches B at time t_2' by B's clock, is then reflected by
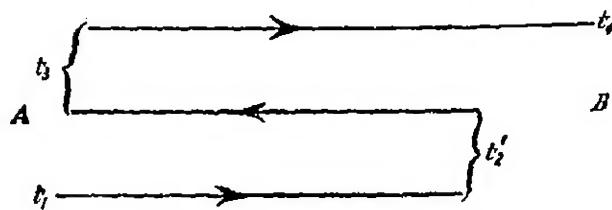
FIG. 1.

B and reaches A at time t_3 by A's clock, and is again reflected and reaches B at time t_4' by B's clock. Then, applying the conventional definitions of coordinates, we have

$$t_B = \tfrac{1}{2}(t_3 + t_1), \qquad r_B = \tfrac{1}{2}c(t_3 - t_1),$$

and

$$t_A' = \tfrac{1}{2}(t_4' + t_2'), \qquad r_A' = \tfrac{1}{2}c(t_4' - t_2'),$$

where (t_B, r_B) are coordinates assigned by A, (t_A', r_A') are coordinates assigned by B.

Eliminating the observed epochs t_1 and t_4' which are found at the beginning and end of the process, we have

$$t_3 = t_B + r_B/c, \qquad t_2' = t_A' - r_A'/c.$$

But

$$r_B = c\phi(t_B), \qquad t_2' = f(t_B),$$

$$r_A' = c\phi(t_A'), \qquad t_3 = f(t_A').$$

Eliminating the observations t_3 and t_2', we have

$$f(t_A') = t_B + \phi(t_B), \tag{20}$$

$$f(t_B) = t_A' - \phi(t_A'). \tag{21}$$

Now t_1 is arbitrary. This implies that either of the coordinates t_B or t_A' is arbitrary. Hence, if t_A' is eliminated between (20) and (21), the result must be an identity in t_B; and similarly if t_B is eliminated, the result must be an identity in t_A'. That is to say, the pair of relations

$$f(x) = y + \phi(y), \qquad f(y) = x - \phi(x), \tag{22}$$

are such that each must be an identity in x or in y when the remaining variable is eliminated. Thus given ϕ, f is determined. Hence

for the most general relative motion of two symmetrically related observers A and B, their clock-running relation is determinate.

22. Relation to signal functions. The second of (22) can be written

$$y = f^{-1}\{x - \phi(x)\}.$$

Substituting in the first of (22), we get

$$f(x) \equiv f^{-1}\{x - \phi(x)\} + \phi f^{-1}\{x - \phi(x)\}.$$

Putting $f^{-1}(x)$ for x, we get

$$x \equiv f^{-1}\{f^{-1}(x) - \phi f^{-1}(x)\} + \phi f^{-1}\{f^{-1}(x) - \phi f^{-1}(x)\}$$

$$\equiv (f^{-1} + \phi f^{-1})(f^{-1} - \phi f^{-1})x.$$

Hence $f^{-1} + \phi f^{-1}$ and $f^{-1} - \phi f^{-1}$ are inverse operators. Accordingly we may put

$$f^{-1} + \phi f^{-1} = p, \tag{23}$$

$$f^{-1} - \phi f^{-1} = p^{-1}. \tag{23'}$$

Adding and subtracting,

$$f^{-1} = \tfrac{1}{2}(p + p^{-1}), \tag{24}$$

$$\phi f^{-1} = \tfrac{1}{2}(p - p^{-1}). \tag{25}$$

Then since we have written x for t'_A, y for t_B, we have

$$t'_2 = f(t_B) = f(y) = x - \phi(x) = t'_A - \phi(t'_A) = f^{-1}(t_3) - \phi f^{-1}(t_3) = p^{-1}(t_3),$$

and likewise

$$t_1 = t_B - \phi(t_B) = f^{-1}(t'_2) - \phi f^{-1}(t'_2) = p^{-1}(t'_2),$$

and

$$t'_4 = t'_A + \phi(t'_A) = f^{-1}(t_3) + \phi f^{-1}(t_3) = p(t_3).$$

Thus

$$t'_2 = p(t_1),$$
$$t_3 = p(t'_2),$$
$$t'_4 = p(t_3), \tag{26}$$

and therefore the function $p(t)$ is precisely the signal function $\theta_{18}(t)$ connecting observers A and B. Thus, given θ we can determine in turn f and ϕ.

23. Transformation formulae. Consider next the relations between A's and B's assignments of coordinates to a distant collinear event E. Let a light-signal leave A at epoch t_1 by A's clock, pass over B at epoch t'_2 by B's clock, reach a distant particle P collinear with A and B, be reflected at P, return to B at epoch t'_3 by B's clock, and finally reach A at time t_4 by A's clock. Let E be the event of

reflection at P, and let (t,x) be the coordinates assigned by A to E as a result of his (A's) observations, (t',x') the coordinates assigned by B to E as a result of his (B's) observations. Then by the definitions of the coordinates

$$t = \tfrac{1}{2}(t_4+t_1), \qquad x = \tfrac{1}{2}c(t_4-t_1),$$
$$t' = \tfrac{1}{2}(t_3'+t_2'), \qquad x' = \tfrac{1}{2}c(t_3'-t_2').$$

Hence
$$t_1 = t-x/c, \qquad t_4 = t+x/c,$$
$$t_2' = t'-x'/c, \qquad t_3' = t'+x'/c.$$

But
$$t_2' = \theta_{12}(t_1),$$
$$t_4 = \theta_{12}(t_3').$$
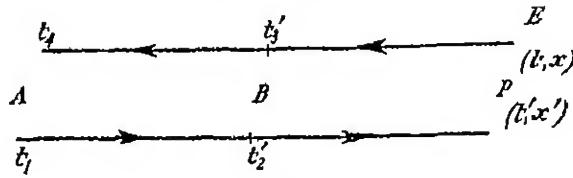
Fig. 2.

Hence
$$t'-x'/c = \theta_{12}(t-x/c), \tag{27}$$
$$t'+x'/c = \theta_{21}(t+x/c), \tag{28}$$

where as usual $\theta_{21} \equiv \theta_{12}^{-1}$. These are a pair of simultaneous equations for t', x' in terms of t, x, given the signal function θ_{12} connecting the observers A and B. Relations (27) and (28) hold good whatever the relative motion of A and B. The relative motion of A and B is expressed by the function θ, as we shall now see.

24. Examples of linear equivalences and corresponding transformation formulae. (1) The simplest possible generating function of a linear equivalence is given by $\psi(t) \equiv t$. Then $\psi^{-1}(t) \equiv t$ and

$$\theta_{pq}(t) = \psi\alpha_{pq}\psi^{-1}(t) = \alpha_{pq}t. \tag{29}$$

Hence, by (24),
$$f^{-1}(t) = \tfrac{1}{2}\{\theta_{pq}(t)+\theta_{qp}(t)\}$$
$$= \tfrac{1}{2}(\alpha_{pq}+\alpha_{pq}^{-1})t.$$

Hence
$$f(t) = \frac{2t}{\alpha_{pq}+\alpha_{pq}^{-1}}. \tag{30}$$

Hence, by (25),
$$\phi\big(\tfrac{1}{2}(\alpha_{pq}+\alpha_{pq}^{-1})t\big) = \tfrac{1}{2}(\alpha_{pq}-\alpha_{pq}^{-1})t,$$

or
$$\phi(t) = \frac{\alpha_{pq}-\alpha_{pq}^{-1}}{\alpha_{pq}+\alpha_{pq}^{-1}}\,t = \frac{\alpha_{pq}^2-1}{\alpha_{pq}^2+1}\,t.$$

Thus the epoch-distance relation for this equivalence is

$$c\phi(t) = c\frac{\alpha_{pq}^2-1}{\alpha_{pq}^2+1}t. \tag{31}$$

This shows that to A, B is moving with a certain uniform velocity, namely, the constant $c\phi'(t)$. Call this V_{pq}. Then

$$\frac{\alpha_{pq}^2-1}{\alpha_{pq}^2+1} = \frac{V_{pq}}{c}, \tag{32}$$

or

$$\alpha_{pq} = \left(\frac{1+V_{pq}/c}{1-V_{pq}/c}\right)^{\frac{1}{2}}, \tag{33}$$

whence with

$$c\phi(t) = V_{pq}t, \tag{34}$$

we have

$$f(t) = (1-V_{pq}^2/c^2)^{\frac{1}{2}}t. \tag{35}$$

The linear equivalence determined by $\psi(t) \equiv t$ is thus the aggregate of all particle-observers moving with relative uniform velocity and separating from one another at a common epoch $t = 0$. The form of $f(t)$ shows that any member of the equivalence reckons the clock of any other member as running slow. We call this equivalence the 'uniform motion equivalence'.

The transformation formulae corresponding to the uniform motion equivalence are, by (27), (28), and (33),

$$t'-x'/c = \left(\frac{1+V_{12}/c}{1-V_{12}/c}\right)^{\frac{1}{2}}(t-x/c), \tag{36}$$

$$t'+x'/c = \left(\frac{1-V_{12}/c}{1+V_{12}/c}\right)^{\frac{1}{2}}(t+x/c), \tag{37}$$

which yield on solution for t' and x'

$$t' = \frac{t-V_{12}x/c^2}{(1-V_{12}^2/c^2)^{\frac{1}{2}}}, \tag{38}$$

$$x' = \frac{x-V_{12}t}{(1-V_{12}^2/c^2)^{\frac{1}{2}}}. \tag{39}$$

These are the famous Lorentz formulae for the transformation of the coordinates (t,x) observed by an observer A into the coordinates (t',x') observed by a second observer B moving with uniform velocity V_{12} with respect to A. Moreover, since for three observers we have the relation

$$\alpha_{pr} = \alpha_{pq}\alpha_{qr},$$

it follows from (33) that

$$\frac{1+V_{pr}/c}{1-V_{pr}/c} = \frac{1+V_{pq}/c}{1-V_{pq}/c}\frac{1+V_{qr}/c}{1-V_{qr}/c}. \tag{40}$$

This is one of the forms of Einstein's relative-velocity formulae. It gives

$$V_{pr} = \frac{V_{pq} + V_{qr}}{1 + V_{pq}V_{qr}/c^2},\tag{41}$$

whence

$$V_{qr} = \frac{V_{pr} - V_{pq}}{1 - V_{pq}V_{pr}/c^2}.\tag{42}$$

The latter gives the relative velocity of A_r and A_q, as observed by A_q, in terms of the velocities of A_r and A_q observed by A_p.

It is worth while to pause for a moment to consider what has gone to the establishment of these formulae. They appear as the conditions that two observers A and B make observations of a collinear event E which are consistent with the observations that A and B can make on one another, when the relation of A to B is symmetrical. The only types of observation that it has been necessary to assume are the actual perceptions of the event E by A and by B, and the perception of A and B by one another. No quantitative proporties of light have been assumed whatever. The only physical property of light assumed is that when A strikes a light at himself he sees the distant illuminated object (the distant clock) *after* the moment of striking the light. Put in one way, this is equivalent to the assumption of the finiteness of the velocity of propagation of light. But strictly speaking, the latter is a deduction from the more primitive property stated in the last sentence but one, and this in turn may be regarded as a consequence of a primitive type of axiom of causation, namely, that the *effect* (namely, the seeing by A of B's clock) is an event in A's experience which must be after and not before its *cause* (namely, the striking of a light by A). Without a regularity of this primitive kind, the world would be a very topsy-turvy one indeed. I repeat that only the most primitive elements of perception have gone to the establishment of the Lorentz formulae; and they are in fact the expression of the analysis of the act of perception into its elements. We have not found it necessary to assume the constancy of the speed of light, though this is an *a posteriori* consequence of our analysis.

The Lorentz formulae will be used freely in later developments in this volume. They will always be used in the sense in which they have been established, namely, as formulae of transformation between the members of the uniform-relative-motion equivalence $\psi(t) \equiv t$.

25. If we take the next simplest linear equivalence, generated by $\psi(t) \equiv A t^s / t_0^{s-1}$, we get nothing new. For the signal functions, using $\psi^{-1}(t) \equiv (t t_0^{s-1}/A)^{1/s}$, come out to be

$$\theta_{pq}(t) \equiv \alpha_{pq}^s \, t,$$

and so are the same totality of functions as for $\psi(t) \equiv t$. The relative motion between members of the equivalence is again one of uniform velocity.

26. Second example of an equivalence. (2) Consider the equivalence generated by $\psi(t) = t_0 \log(t/t_0)$. (We introduce an arbitrary parameter t_0, of the same physical dimensions as t, so that $\psi(t)$ remains of the dimensions of an epoch.) Then $\psi^{-1}(t) = t_0 e^{t/t_0}$, and

$$\theta_{pq}(t) = t + t_0 \log \alpha_{pq}, \tag{43}$$

$$\theta_{qp}(t) = t - t_0 \log \alpha_{pq}. \tag{44}$$

Hence

$$f^{-1}(t) = \tfrac{1}{2}\{\theta_{pq}(t) + \theta_{pq}^{-1}(t)\} \equiv t,$$

and so

$$f(t) \equiv t. \tag{45}$$

Then

$$\phi f^{-1}(t) = \tfrac{1}{2}\{\theta_{pq}(t) - \theta_{pq}^{-1}(t)\} = t_0 \log \alpha_{pq},$$

or

$$c\phi(t) = c t_0 \log \alpha_{pq} = \text{const.} \tag{46}$$

Thus in this equivalence, the members are relatively stationary, and the relation $f(t) \equiv t$ shows that there is an absolute simultaneity amongst all the clock members of the equivalence. The transformation formulae give

$$t' + x'/c = \theta_{qp}(t + x/c) = t + x/c - t_0 \log \alpha_{pq},$$

$$t' - x'/c = \theta_{pq}(t - x/c) = t - x/c + t_0 \log \alpha_{pq},$$

whence

$$t' = t, \tag{47}$$

$$x' = x - c t_0 \log \alpha_{pq}. \tag{48}$$

The parameter α_{pq} is thus a measure of the separation of the relatively stationary members A_p, A_q, of the equivalence, and the relation $t' = t$ again shows that there is an absolute simultaneity, all members of the equivalence attaching the same epoch to any event.

27. By the general theory it must be possible to find a regraduation of the clocks of the uniform-relative-motion equivalence which connects them with the stationary equivalence. By formula (14), a ϕ-equivalence $\phi \equiv t$ is transformed into the ψ-equivalence

$\psi \equiv t_0 \log t/t_0$ if we renumber the clock-reading t to read τ, where $\tau = \chi(t)$ and χ is given by

$$\chi(t) \equiv \psi[k\{\phi^{-1}(t)\}^s]$$
$$\equiv t_0 \log(kt^s/t_0)$$
$$= st_0 \log(t/t_0) + \text{const.}$$

Choice of s and k correspond to an arbitrary choice of zero and scale-factor in τ. We may consider in particular the regraduation of clocks

$$\tau = t_0 \log(t/t_0) + t_0, \tag{49}$$

which makes $\tau = t_0$ and $d\tau/dt = 1$ at $t = t_0$. The signal function $\theta_{pq}(t) = \alpha_{pq} t$, implying the signal relation

$$t'_2 = \alpha_{pq} t_1,$$

becomes on regraduation

$$t_0 e^{(\tau'_2/t_0)} = t_0 \alpha_{pq} e^{(\tau_1/t_0)},$$

or

$$\tau'_2 = \tau_1 + t_0 \log \alpha_{pq},$$

and the new signal function $\vartheta_{pq}(\tau)$ is

$$\vartheta_{pq}(\tau) = \tau + t_0 \log \alpha_{pq}, \tag{50}$$

which as before generates the relatively stationary equivalence.

28. An accelerated equivalence. (3) Any number of other equivalences may be generated by suitable choice of generating functions. It has been shown by Whitrow and the writer[†] that an equivalence in which relatively accelerated members occur, discovered by Leigh Page[‡], can be generated by the function

$$\psi(t) \equiv \frac{t_0}{\log(t_0/t)}, \qquad \psi^{-1}(t) = t_0 e^{-t_0/t} \qquad (t_0 > 0).$$

The signal functions are given by

$$\theta_{pq}(t) = \frac{t}{1 - (t/t_0) \log \alpha_{pq}}.$$

To examine the relative motions in this equivalence, put

$$\log \alpha_{pq} = a_{pq},$$

so that

$$a_{qp} = -a_{pq},$$

and

$$\theta_{pq}(t) = \frac{t}{1 - (t/t_0) a_{pq}}.$$

† *Zeits. für Astrophys.* **15**, 342, 1938.
‡ L. Page, *Phys. Review*, **49**, 254, 466, 1936.

Thus, in the usual notation, if a signal leaves A_p at time t_1 by A_p's clock, reaches A_q at time t_2' by A_q's clock, and returns to A_p at time t_3 by A_p's clock, then

$$t_2' = \frac{t_1}{1-(t_1/t_0)a_{pq}}, \qquad t_3 = \frac{t_2'}{1-(t_2'/t_0)a_{pq}}.$$

Hence

$$\frac{1}{t_1}-\frac{1}{t_2'} = \frac{a_{pq}}{t_0}, \qquad \frac{1}{t_2'}-\frac{1}{t_3} = \frac{a_{pq}}{t_0},$$

whence

$$\frac{1}{t_1}-\frac{1}{t_3} = \frac{2a_{pq}}{t_0}.$$

Since $t_1 < t_3$, we must have $a_{pq} > 0$. To determine the relative motion, we have if (t, x) are the coordinates assigned by A_p to A_q at the event at A_q which A_q records as at t_2', then

$$t = \tfrac{1}{2}(t_3+t_1), \qquad x = \tfrac{1}{2}c(t_3-t_1),$$

whence

$$\frac{1}{t-x/c}-\frac{1}{t+x/c} = \frac{2a_{pq}}{t_0},$$

and therefore

$$\frac{x^2}{c^2}+\frac{x}{c}\frac{t_0}{a_{pq}}-t^2 = 0.$$

We note that at $x = 0$, $t = 0$, $(dx/dt)_0 = 0$ and thus A_q moves from relative rest at A_p at $t = 0$. Now differentiate the last equation but one, and put $dx/dt = V$. We get

$$\frac{1-V/c}{(t-x/c)^2}-\frac{1+V/c}{(t+x/c)^2} = 0,$$

or

$$\frac{1-V/c}{1+V/c} = \frac{(t-x/c)^2}{(t+x/c)^2} = \left(\frac{t_1}{t_3}\right)^2.$$

But

$$dt_1 = (1-V/c)\,dt, \qquad dt_3 = (1+V/c)\,dt.$$

Differentiating the last equation but one to give the acceleration, we get

$$-\frac{dV}{1-V^2/c^2} = c\left(\frac{dt_1}{t_1}-\frac{dt_3}{t_3}\right) = c\left(\frac{1-V/c}{t_1}-\frac{1+V/c}{t_3}\right)dt,$$

whence

$$\frac{1}{(1-V^2/c^2)^{\frac{3}{2}}}\frac{dV}{dt} = c\left\{-\left(\frac{1-V/c}{1+V/c}\right)^{\frac{1}{2}}\frac{1}{t_1}+\left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}\frac{1}{t_3}\right\}$$

$$= c\left\{\frac{1}{t_1}-\frac{1}{t_3}\right\} = 2c\frac{a_{pq}}{t_0} = \text{const.}$$

Thus the initial acceleration of A_q is $2ca_{pq}/t_0$; the acceleration

decreases as V increases, i.e. as t increases. The paper cited investigates several curious properties of this equivalence.

29. A non-intersecting accelerated equivalence. (4) An equivalence of this description, in which the members never coincide with one another, is given by

$$\psi(t) = (t^2 - t_0^2)^{\frac{1}{2}}, \qquad \theta_{pq} = \{a_{pq}^2 t^2 + t_0^2(\alpha_{pq}^2 - 1)\}^{\frac{1}{2}}.$$

This may be investigated by the reader.

30. The two scales of time. By the main theorem on equivalences, all the apparently different equivalences generated by different functions ψ are so many different descriptions of that unique entity, the kinematic equivalence. The apparent differences arise from the different possible ways in which the 'clocks' used by the particle-members of the equivalence may be graduated. In particular, clock graduations can be found so that an equivalence is described as consisting of particles in uniform relative motion separating from a point of common coincidence; and clock graduations can be found so that the same equivalence appears to consist of relatively stationary particles. We can thus use an equivalence to isolate two measures of time, t and τ, one of which (t) is indeterminate to a multiple of a mononomial power, the other (τ) of which is indeterminate to a change of scale and origin. The question arises whether either of these may be identified with Newtonian time, the time of physics. This, the next problem of time-keeping, will be the subject of Part II. It can only be solved by deducing by kinematic methods the dynamics of a particle.

But since the 'uniform' time of dynamics admits a change of origin and a change of scale without affecting the form of the equations of dynamics, we have a strong suggestion that it will be the time τ that is finally to be identified with Newtonian time. It can be shown, in fact, that the relatively stationary equivalence is the only form of equivalence which isolates a clock whose readings admit of a linear transformation without altering the apparent description of the equivalence.

31. Role of the constant t_0. If, as we are anticipating, the τ-scale of time, or the time which renders the equivalence stationary, is finally to be identified with the 'uniform' time of physics, it might be thought that the τ-measure of time was the more fundamental

of the two considered. This, however, is not so. For it can only be described with the aid of a parameter t_0, which occurs in the generating function of the τ-equivalence, namely $\psi(\tau) \equiv t_0 \log(t/t_0)$. The t-form of the equivalence can be described, on the other hand, by the simple generating function $\psi(t) \equiv t$, from which t_0 is absent. The status of t_0 is not at all evident from the considerations so far advanced. It will be shown later that so long as we confine attention to descriptions using the t-measure of time, no parameter t_0 makes its appearance; but that corresponding descriptions in the τ-measure of time always make mention of t_0. This is connected with the circumstance that the t-equivalence possesses a natural origin of time, $t = 0$, the epoch of coincidence of all its members. The value of t at any event may be called the *age* of the system at that event. From the clock-regraduation formula

$$\tau = t_0 \log(t/t_0) + t_0,$$

it is apparent that $t = 0$ corresponds to $\tau = -\infty$. The zero of time is thus inaccessible on the τ-scale. The times τ and t agree at the epoch $t = t_0$. Consequently, if we want the t- and τ-scales to agree at the present epoch, we must choose t_0 to be the value of the age of the system on the t-scale, reckoned from the natural origin of time.

But there is no unforced way of introducing t_0 if we begin with the relatively stationary equivalence and τ-time, for there is no natural origin of time on this scale. This will become clear when we come to construct equations of motion. These can be constructed in t-measures with ease, and accordingly we shall first construct a dynamics in t-measure. It will prove to be very different from Newtonian dynamics. But it will be shown to pass into Newtonian dynamics on regraduating clocks from t to τ.

32. Velocity-distance relations. We conclude this chapter by showing how to deduce the velocity-distance relationship for any equivalence, generated by an arbitrary positive monotonic function ψ possessing a unique inverse. Since, by (22) and (23),

$$f^{-1}(t) + \phi f^{-1}(t) = \theta(t), \tag{51}$$

$$f^{-1}(t) - \phi f^{-1}(t) = \theta^{-1}(t), \tag{52}$$

we have

$$t + \phi(t) = \theta f(t), \tag{53}$$

$$t - \phi(t) = \theta^{-1} f(t). \tag{54}$$

Now let x denote the distance-coordinate at time t of the second of the two observers connected by the signal function $\theta(t)$. Then $x = c\phi(t)$. Hence

$$t + x/c = \psi\alpha\psi^{-1}f(t), \tag{55}$$

$$t - x/c = \psi\alpha^{-1}\psi^{-1}f(t). \tag{56}$$

Hence

$$\alpha\psi^{-1}f(t) = \psi^{-1}(t + x/c),$$

$$\alpha^{-1}\psi^{-1}f(t) = \psi^{-1}(t - x/c).$$

Hence

$$\alpha^2 = \frac{\psi^{-1}(t + x/c)}{\psi^{-1}(t - x/c)}. \tag{57}$$

This is the desired relation between t and x for the pair of observers connected by the parameter α. Again, differentiating (55) and (56), and putting $dx/dt = v$,

$$1 + v/c = \alpha\psi^{-1\prime}f(t)\psi'\alpha\psi^{-1}f(t)f'(t),$$

$$1 - v/c = \alpha^{-1}\psi^{-1\prime}f(t)\psi'\alpha^{-1}\psi^{-1}f(t)f'(t),$$

whence

$$\frac{1 + v/c}{1 - v/c} = \frac{\psi'\alpha\psi^{-1}f(t)}{\psi'\alpha^{-1}\psi^{-1}f(t)}\alpha^2. \tag{58}$$

Using (55), (56), and (57) to eliminate α from (58), we have

$$\frac{1 + v/c}{1 - v/c} = \frac{\psi'\psi^{-1}(t + x/c)}{\psi'\psi^{-1}(t - x/c)}\frac{\psi^{-1}(t + x/c)}{\psi^{-1}(t - x/c)}.$$

This gives the velocity v in the equivalence ψ in terms of the distance x of the particle at epoch t.

33. Examples. (1) The uniform motion equivalence, $\psi(t) \equiv t$. Then

$$\frac{1 + v/c}{1 - v/c} = \frac{t + x/c}{t - x/c},$$

or

$$v = x/t.$$

(2) $\quad \psi(t) = At^s/t_0^{s-1}, \quad \psi^{-1}(t) = (tt_0^{s-1}/A)^{1/s}, \quad \psi'(t) = sA(t/t_0)^{s-1}.$

Then

$$\frac{1 + v/c}{1 - v/c} = \frac{t + x/c}{t - x/c},$$

or

$$v = x/t,$$

as in (1).

(3) Take

$$\psi(t) = t_0 \log(t/t_0) + b.$$

Then

$$\psi^{-1}(t) = t_0 e^{(t-b)/t_0}, \qquad \psi'\psi^{-1}(t) = e^{-(t-b)/t_0},$$

and so

$$\frac{1 + v/c}{1 - v/c} = 1,$$

or

$$v = 0.$$

These results verify the earlier theory.

THE THREE-DIMENSIONAL EQUIVALENCE

34. Generalization to three dimensions. The idea of a linear equivalence is readily generalized to three dimensions. Take a pencil of linear equivalences through a particle O, all the linear equivalences being generated by the same function ψ, and reduce them all to uniform relative motion by suitable regraduation of O's clock and the same regraduation of all the other clocks carried by the members of the equivalences. Take a member A_p of one of these equivalences. Then the particle-observers on any straight line through A_p are also in uniform relative motion, and so constitute an equivalence, whence A_p is also the vertex of a set of uniform relative motion equivalences; A_p's relation to the members of any linear set of particle-observers through A_p is indistinguishable from O's. Now regraduate back again, recovering the ψ-equivalences. In this way the private three-dimensional space of any member A_p of any of the original set of equivalences is populated with equivalent particles possessing relative motions compatible with their being equipped with congruent clocks. We may call the resulting set of particle-observers a three-dimensional equivalence, or, more briefly, an equivalence.

It must be emphasized that in an equivalence we pay attention only to the nature of the relative motions, not to the density-distribution of particles. That will come in later, as we advance from kinematics to dynamics. So far we have been considering *prescribed* motions; we have not yet considered how they can originate.

35. Transformation formulae in general. But before we advance to dynamics, we should consider the general transformation of the coordinates of an event, and not merely the transformation (27), (28) of the previous chapter which is concerned only with an event collinear with the two observers in question.

Let E be an event, A a given observer, a member of a general equivalence. Choose an observer B, a member of the equivalence, near the event E. Then A and B will have a certain relative motion of approach or recession.

Let a light-signal leave A at epoch t_1 by A's clock so as to exhibit to A the event E in question; let A then see E at epoch t_4, i.e. t_4 is the epoch of return to A of the signal reflected at E. Similarly let

a signal leave B at epoch t_2' by B's clock, be reflected at the same event E, and return to B at epoch t_3' by B's clock. Let t_2 and t_3 be the epochs assigned by A to the events at B which were recorded by B as occurring at B at epochs t_2' and t_3'. Let x, t be the distance and epoch assigned by A to the event E. Then

$$x = \tfrac{1}{2}c(t_4-t_1), \qquad t = \tfrac{1}{2}(t_4+t_1),$$

so that
$$t - x/c = t_1, \qquad t + x/c = t_4. \qquad\qquad (1),\ (2)$$

Since A knows B to be 'near' E, A can regard x, the distance of E from A, as equal approximately to the projected distance of E on the line AB.

A and B now make the following diagrams, A's diagram being constructed as though he (A) were at rest, B's diagram being con-
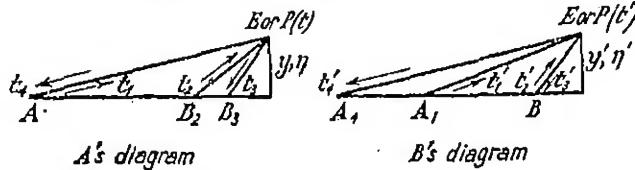
Fig. 3.

structed as though he (B) were at rest. A now defines the transverse coordinate y of E as derived from A's account of B's observation t_2 by the relation
$$(x-x_2)^2+y^2 = c^2(t-t_2)^2, \qquad\qquad (3)$$

and he gets an alternative coordinate η from B's observation t_3 by the relation
$$(x-x_3)^2+\eta^2 = c^2(t-t_3)^2. \qquad\qquad (4)$$

(We shall show later that $\eta = y$.)

Now consider B's diagram of the state of affairs. B, from his knowledge of A's motion relative to him, can calculate the epochs t_1' and t_4' which he would attach to the epochs of the events at A to which A gives the epochs t_1 and t_4; and B knows the distances x_1' and x_4' he would attach to the events. Accordingly B defines coordinates x' and t' of the event E (to himself) by means of the relations

$$x'+x_1' = c(t'-t_1'), \qquad\qquad (5)$$

$$x'+x_4' = c(t_4'-t'). \qquad\qquad (6)$$

Observer B further defines small members y' and η' by the relations

$$x'^2+y'^2 = c^2(t'-t_2')^2, \qquad\qquad (7)$$

$$x'^2+\eta'^2 = c^2(t_3'-t')^2. \qquad\qquad (8)$$

But since (t'_1, x'_1) are the assignments by B of coordinates which to A are $(t_1, 0)$, we have, by the one-dimensional transformation formulae of the previous chapter,

$$t'_1 + x'_1/c = \theta_{12}(t_1). \tag{9}$$

Similarly,
$$t'_4 - x'_4/c = \theta_{21}(t_4). \tag{10}$$

But, by (5),
$$t'_1 + x'_1/c = t' - x'/c, \tag{11}$$

and similarly, by (6),
$$t'_4 - x'_4/c = t' + x'/c. \tag{12}$$

Hence
$$t' - x'/c = \theta_{12}(t_1) = \theta_{12}(t - x/c), \tag{13}$$

by (1); and similarly

$$t' + x'/c = \theta_{21}(t_4) = \theta_{21}(t + x/c). \tag{14}$$

Thus the *form* of the general transformation formulae (27), (28) of the previous chapter persists with our definition of the 'small' coordinate x' of E.

We wish to determine the corresponding transformation formulae connecting y and y', or η and η'. By (3),

$$\frac{y^2}{c^2} = (t - t_2)^2 - \frac{(x - x_2)^2}{c^2} = \left\{ \left(t + \frac{x}{c} \right) - \left(t_2 + \frac{x_2}{c} \right) \right\} \left\{ \left(t - \frac{x}{c} \right) - \left(t_2 - \frac{x_2}{c} \right) \right\}. \tag{15}$$

From this we have to eliminate x_2 and t_2 in terms of coordinates assigned by B. By (7), and the definitions of signal functions,

$$t_2 + \frac{x_2}{c} = \theta_{12}(t'_2 + 0) = \theta_{12} \left\{ t' - \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\},$$

and
$$t_2 - \frac{x_2}{c} = \theta_{21}(t'_2 - 0) = \theta_{21} \left\{ t' - \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\}.$$

Hence, from (15),

$$\frac{y^2}{c^2} = \left[\theta_{12} \left(t' + \frac{x'}{c} \right) - \theta_{12} \left\{ t' - \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\} \right] \times$$

$$\times \left[\theta_{21} \left(t' - \frac{x'}{c} \right) - \theta_{21} \left\{ t' - \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\} \right].$$

But x' and y' are small numbers. Hence, using Taylor's theorem, we have

$$\frac{y^2}{c^2} = \theta'_{12}(t') \left\{ \frac{x'}{c} + \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\} \theta'_{21}(t') \left\{ -\frac{x'}{c} + \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{c} \right\},$$

i.e.
$$y = \{\theta'_{12}(t') \theta'_{21}(t')\}^{\frac{1}{2}} y'. \tag{16}$$

If the same calculations are pursued with η and η', we get by similar processes
$$\eta = \{\theta'_{12}(t')\theta'_{21}(t')\}^{\frac{1}{2}}\eta'. \tag{17}$$

Now we wish to have t' equal to $\frac{1}{2}(t'_2 + t'_3)$. Hence (7) and (8) require $y' = \eta'$. Hence $y = \eta$.

36. Summary of transformation formulae.

Summarizing, we have the set of transformation formulae in the form
$$t' - x'/c = \theta_{12}(t - x/c), \tag{18}$$
$$t' + x'/c = \theta_{21}(t + x/c), \tag{19}$$
$$y = \{\theta'_{12}(t')\theta'_{21}(t')\}^{\frac{1}{2}}y', \tag{20}$$
where t', x', y' are the coordinates assigned to E by an observer B near E.

In general we see that the lateral coordinate y is altered on transformation from one member of the equivalence to another. But in the case of the uniform motion equivalence, where
$$\theta_{12}(t') = \left(\frac{1 + V_{12}/c}{1 - V_{12}/c}\right)^{\frac{1}{2}}t', \qquad \theta_{21}(t') = \left(\frac{1 - V_{12}/c}{1 + V_{12}/c}\right)^{\frac{1}{2}}t',$$

we see that $y = y'$. This is the transverse formula in the Lorentz transformation. It is easily seen that in this case, the relation $y = y'$ holds for finite y, since the transformation functions θ_{12} and θ_{21} are linear.

37. Passage from one arbitrary observer to another.

Having now passed from A to an observer B near E, we can readily pass from any observer O, a member of the equivalence, to any other observer O', a member of the same equivalence.

Let O, a member of the equivalence, observe two neighbouring events E, E_1, and let O' (a member of the same equivalence) be passing through E at the epoch of occurrence of E. For O, the coordinates of E are to be (t, x), for O', $(t', 0)$. Let the event E_1 be $(t + dt, x + dx, y)$ for O, and $(t' + dt', x', y')$ to O'. Then by the transformation formulae established,
$$t + x/c = \theta_{12}(t'), \qquad t - x/c = \theta_{21}(t'),$$
$$t + dt + \frac{x + dx}{c} = \theta_{12}\left(t' + dt' + \frac{x'}{c}\right), \qquad t + dt - \frac{x + dx}{c} = \theta_{21}\left(t' + dt' - \frac{x'}{c}\right),$$
and
$$y^2 = \theta'_{12}(t')\theta'_{21}(t')y'^2.$$

From these

$$dt + \frac{dx}{c} = \left(dt' + \frac{x'}{c}\right)\theta'_{12}(t'), \qquad dt - \frac{dx}{c} = \left(dt' - \frac{x'}{c}\right)\theta'_{21}(t').$$

Hence

$$dt^2 - \frac{dx^2}{c^2} = \theta'_{12}(t')\theta'_{21}(t')\left(dt'^2 - \frac{x'^2}{c^2}\right),$$

and thus, in all,

$$\frac{dt^2 - (dx^2 + y^2)/c^2}{\theta'_{12}(t')\theta'_{21}(t')} = dt'^2 - \frac{(x'^2 + y'^2)}{c^2}. \tag{21}$$

Now t' is the epoch of E observed by O' at E, and $t' + dt'$ is equal, in the notation of the previous section, to $\frac{1}{2}(t'_3 + t'_2)$, and so also depends on observations made by O'. Further, by (7) and (8) above, $(x'^2 + y'^2)$ depends solely on O''s measures. Hence the right-hand side of (21) depends solely on O'. Hence the left-hand side must be the same for all observers O. (It should be remembered that the function θ_{12} and its inverse θ_{21} are different for every different O, being the functions describing the motion of each O with respect to O'.)

The small number $(dx^2 + y^2)$ may be called the square of the spatial separation of the events E, E_1, in O's private Euclidean space. Calling it de^2, we have that

$$ds_1^2 = \frac{dt^2 - de^2/c^2}{\theta'_{12}(t')\theta'_{21}(t')}$$

is the same for all observers O, members of the equivalence.

We now want to express ds_1^2 in terms of coordinates used by O alone and the generating function ψ of the equivalence. Calling 'x' now 'r', the distance of E from O, we have

$$t + r/c = \theta_{12}(t' + 0) = \psi\alpha_{12}\psi^{-1}(t'),$$
$$t - r/c = \theta_{21}(t') = \psi\alpha_{12}^{-1}\psi^{-1}(t').$$

These two relations determine α_{12} and t' in terms of t and r, which are coordinates used by O. Eliminating α_{12}, we have

$$\psi^{-1}(t + r/c)\psi^{-1}(t - r/c) = \{\psi^{-1}(t')\}^2.$$

This determines t'. Now

$$\theta'_{12}(t')\theta'_{21}(t') = \{\psi'\alpha_{12}\psi^{-1}(t')\psi'\alpha_{12}^{-1}\psi^{-1}(t')\}\{\psi^{-1\prime}(t')\}^2,$$

and

$$\psi^{-1\prime}(t') \equiv 1/\psi'\psi^{-1}(t').$$

Hence altogether

$$ds_1^2 = (dt^2 - de^2/c^2)\frac{[\psi'\{\psi^{-1}(t + r/c)\psi^{-1}(t - r/c)\}^{\frac{1}{2}}]^2}{\psi'\psi^{-1}(t + r/c)\psi'\psi^{-1}(t - r/c)}, \tag{22}$$

and this is the same in form and value for all members O of the equivalence. Moreover, its value is equal to $dt'^2 - de'^2/c^2$.

The number t' is an invariant, the same for all O's. Hence ds_1^2 could be multiplied by any function of t', and still retain its property of being unaltered in form and value for all O's. Let us therefore attempt to determine a multiplier of ds_1^2 such that after multiplication its new value ds^2 is invariant under any regraduation of the clocks of the equivalence.

38. Regraduation of clocks. As in the preceding chapter, let all observers O, members of the equivalence, furnished as they are with congruent clocks, regraduate these clocks from t to T, where $t = \chi(T)$. Under this regraduation, let the ψ-equivalence become a Ψ-equivalence. Then as before, for the pair of signals $t_1 \to t_2'$, $t_2' \to t_3$, between any two members of the equivalence, we have

$$t_2' = \theta_{12}(t_1) = \psi\alpha_{12}\psi^{-1}(t_1),$$

$$t_3 = \theta_{12}(t_2') = \psi\alpha_{12}\psi^{-1}(t_2'),$$

and thus

$$T_2' = \chi^{-1}(t_2') = \chi^{-1}\psi\alpha_{12}\psi^{-1}\chi(T_1'),$$

$$T_3 = \chi^{-1}(t_3) = \chi^{-1}\psi\alpha_{12}\psi^{-1}\chi(T_2').$$

We may thus take $\qquad \Psi = \chi^{-1}\psi,$

or $\qquad\qquad\qquad \chi = \psi\Psi^{-1}.$

In the case of the observer O' at the event E, to which he assigns epoch coordinate t', t' is an actual clock-reading, and so becomes on regraduation T', where

$$t' = \chi(T') = \psi\Psi^{-1}(T').$$

On the other hand, O's coordinates for E, namely (t, r), become now coordinates (T, R), where

$$t + r/c = \chi(T + R/c), \qquad t - r/c = \chi(T - R/c). \tag{23}$$

We can now construct the following short table for O''s measures of the coordinates of the events E and E_1:

Type of coordinate	t	r	T	R
Event E	t'	0	T'	0
Event E_1	$t' + dt'$	r'	$T' + dT'$	R'

Formulae (23), being simply regraduation formulae, can be applied

to O''s clock-readings of the type $t' \pm r'/c$, $T' \pm R'/c$, and so, applying them to the entries occurring in the table,

$$t' + dt' + r'/c = \chi(T' + dT' + R'/c),$$
$$t' + dt' - r'/c = \chi(T' + dT' - R'/c),$$

whence using $t' = \chi(T')$ we get, on approximating and multiplying,

$$dt'^2 - r'^2/c^2 = (dT'^2 - R'^2/c^2)\{\chi'(T')\}^2.$$

The differential coefficient $\chi'(T')$ is given by

$$\chi'(T') = \psi'\Psi^{-1}(T')\Psi'^{-1'}(T'),$$

where

$$\Psi'^{-1'}(T') = 1/\Psi'\Psi^{-1}(T').$$

Hence

$$\chi'(T') = \frac{\psi'\Psi^{-1}(T')}{\Psi'\Psi^{-1}(T')} = \frac{\psi'\psi^{-1}(t')}{\Psi'\Psi^{-1}(T')}.$$

Hence

$$\frac{dt'^2 - r'^2/c^2}{\{\psi'\psi^{-1}(t')\}^2} = \frac{dT'^2 - R'^2/c^2}{\{\Psi'\Psi^{-1}(T')\}^2}.$$

But r', R' are the (small) spatial distances between E and E'. Hence we can write the last formula in the form

$$\frac{dt'^2 - de'^2/c^2}{\{\psi'\psi^{-1}(t')\}^2} = \frac{dT'^2 - dE'^2/c^2}{\{\Psi'\Psi^{-1}(T')\}^2}.$$

Rewriting (22), we can put it in the form

$$ds_1^2 = dt'^2 - de'^2/c^2 = (dt^2 - de^2/c^2)\frac{\{\psi'\psi^{-1}(t')\}^2}{\psi'\psi^{-1}(t + r/c)\psi'\psi^{-1}(t - r/c)},$$

and so similarly, in the Ψ-equivalence

$$dS_1^2 = dT'^2 - dE'^2/c^2$$
$$= (dT^2 - dE^2/c^2)\frac{\{\Psi'\Psi^{-1}(T')\}^2}{\Psi'\Psi^{-1}(T + R/c)\Psi'\Psi^{-1}(T - R/c)}.$$

The last three formulae now show that

$$\frac{ds_1^2}{\{\psi'\psi^{-1}(t')\}^2} = \frac{dt'^2 - de'^2/c^2}{\{\psi'\psi^{-1}(t')\}^2} = \frac{dt^2 - de^2/c^2}{\psi'\psi^{-1}(t + r/c)\psi'\psi^{-1}(t - r/c)}$$
$$= \frac{dS_1^2}{\{\Psi'\Psi^{-1}(T')\}^2} = \frac{dT'^2 - dE'^2/c^2}{\{\Psi'\Psi^{-1}(T')\}^2} = \frac{dT^2 - dE^2/c^2}{\Psi'\Psi^{-1}(T + R/c)\Psi'\Psi^{-1}(T - R/c)}.$$

Call the common value of these fractions ds^2. Then it follows that the expression

$$ds^2 = \frac{dt^2 - de^2/c^2}{\psi'\psi^{-1}(t - r/c)\psi'\psi^{-1}(t + r/c)}$$
$$= \frac{dT^2 - dE^2/c^2}{\Psi'\Psi^{-1}(T - R/c)\Psi'\Psi^{-1}(T + R/c)} \tag{24}$$

takes the same value and the same form, for any two given neighbouring events, for all observers O, members of the equivalence and for all modes of graduating their clocks.

39. Choice of metric. We can call this ds^2 the (squared) interval between the events E and E_1; and, from the property we have established for it, it can be adopted as the metric of space-time for all observers O. It is a *public* space-time. We have established the invariance of form and value of this ds^2 purely from the definition and properties of the three-dimensional ψ-equivalence.

The number de^2 is the square of the separation assigned by O, using the t-mode of graduation of his clock, to two neighbouring events counted as simultaneous on O's convention as to simultaneity, i.e. possessing for O the same epoch coordinate. The same holds good of dE^2. But it does not follow that because de^2 has been calculated as if in a private Euclidean space for O, therefore dE^2 is to be calculated as if in a private Euclidean space. The exact position is best explained through examples.

40. Suppose again that a given observer O, a member of the equivalence, regraduates his clock from t to T, where $t = \chi(T)$. Then in the usual way, if he assigns coordinates (t, r), (T, R) to an event on his two distinct modes of clock-graduation, he can write down

$$t - r/c = \chi(T - R/c) = \psi\Psi^{-1}(T - R/c), \qquad (25)$$

$$t + r/c = \chi(T + R/c) = \psi\Psi^{-1}(T + R/c). \qquad (25')$$

From these, by taking differentials of each side and multiplying together,

$$dt^2 - dr^2/c^2 = \chi'(T - R/c)\chi'(T + R/c)(dT^2 - dR^2/c^2). \qquad (26)$$

But
$$\chi'(T - R/c) = \psi'\Psi^{-1}(T - R/c)\Psi^{-1\prime}(T - R/c)$$

$$= \frac{\psi'\psi^{-1}(t - r/c)}{\Psi'\Psi^{-1}(T - R/c)}.$$

Similarly with arguments $(T + R/c)$, $(t + r/c)$. Hence, from (26)

$$\frac{dt^2 - dr^2/c^2}{\psi'\psi^{-1}(t - r/c)\psi'\psi^{-1}(t + r/c)} = \frac{dT^2 - dR^2/c^2}{\Psi'\Psi^{-1}(T - R/c)\Psi'\Psi^{-1}(T + R/c)}. \qquad (27)$$

This relation, unlike (24), is merely the result of the application of the differential calculus to the definitions of coordinates in terms of clock readings. Relation (27) refers only to a single observer; relation

(24) refers to any pair of observers. Now combine (24) and (27), by eliminating dt^2 and dT^2. Then we get

$$\frac{de^2-dr^2}{\psi'\psi^{-1}(t-r/c)\psi'\psi^{-1}(t+r/c)} = \frac{dE^2-dR^2}{\Psi'\Psi^{-1}(T-R/c)\Psi'\Psi^{-1}(T+R/c)}. \quad (28)$$

This relation connects O's measure of a small transverse distance using t-clocks with his measure of the same transverse distance using T-clocks. The interval de^2 may be calculated by O in terms of any set of spatial coordinates he cares to use; relation (28) then tells him what is the corresponding rule for calculating dE^2 when he has regraduated from t to T.

41. Examples. For example, O may adopt a private Euclidean space, in which case he calculates de^2 according to the rule

$$de^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where θ and ϕ are appropriately defined angular coordinates. Then when O regraduates from t to T, so that his ψ-equivalence becomes the Ψ-equivalence, the metric in the Ψ-equivalence is, by (28),

$$dE^2 = dR^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\frac{\Psi'\Psi^{-1}(T-R/c)\Psi'\Psi^{-1}(T+R/c)}{\psi'\psi(t-r/c)\psi'\psi(t+r/c)}. \quad (29)$$

Here r, t are to be expressed in terms of R, T by (25) and (25'). It is apparent that dE^2 will not in general be Euclidean, and may involve T.

42. Take as a particular example the case where ψ corresponds to the uniform motion equivalence, $\psi(t) \equiv t$, and Ψ corresponds to the relatively stationary equivalence, $\Psi(T) = t_0\log(T/t_0) + t_0$. Then

$$\Psi^{-1}(T) = t_0\, e^{(T-t_0)/t_0}, \qquad \Psi'(T) = t_0/T,$$

$$\Psi'\Psi^{-1}(T) = e^{-(T-t_0)/t_0},$$

$$\Psi'\Psi^{-1}(T-R/c)\Psi'\Psi^{-1}(T+R/c) = e^{-2(T-t_0)/t_0},$$

whilst $\qquad \psi'\psi^{-1}(t-r/c)\psi'\psi^{-1}(t+r/c) = 1.$

The general formula (28) then gives

$$\frac{de^2-dr^2}{1} = \frac{dE^2-dR^2}{e^{-2(T-t_0)/t_0}}. \quad (30)$$

The formulae of transformation of coordinates are

$$t-r/c = t_0\, e^{(T-t_0-R/c)/t_0}, \quad (31)$$

$$t+r/c = t_0\, e^{(T-t_0+R/c)/t_0}. \quad (32)$$

The important point to notice is that O may choose a private Euclidean space either in t-measure or T-measure. If he adopts it in t-measure, then

$$de^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

and then, by (30),

$$dE^2 = dR^2 + e^{-2(T-t_0)/t_0}r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{33}$$

But, by (31) and (32),

$$r = ct_0 \, e^{(T-t_0)/t_0}\sinh R/ct_0. \tag{34}$$

Hence (33) gives

$$dE^2 = dR^2 + (ct_0)^2\sinh^2(R/ct_0)(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{35}$$

Thus dE^2 corresponds to a hyperbolic space. But if on the other hand O adopts a private Euclidean space in T-measure, then

$$dE^2 = dR^2 + R^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

and so, by (30),

$$de^2 = dr^2 + e^{2(T-t_0)/t_0}R^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{36}$$

Now, by (31) and (32),

$$R/ct_0 = \tfrac{1}{2}\log\frac{t+r/c}{t-r/c},$$

and

$$e^{2(T-t_0)/t_0} = (t^2 - r^2/c^2)/t_0^2.$$

Hence (36) gives

$$de^2 = dr^2 + c^2(t^2 - r^2/c^2)\left(\tfrac{1}{2}\log\frac{t+r/c}{t-r/c}\right)^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{37}$$

This is the space that O must use in t-measure if he adopts a private Euclidean space in T-measure. It will be noticed that (37), unlike (35), involves the time-coordinate in the metric. The metric (37) takes the same *form* for all observers O in the equivalence, but its value, for a given pair of events, varies with the observer, just as in the case where O adopts a private Euclidean space in the uniform motion equivalence, $de^2 = dx^2 + dy^2 + dz^2$ takes the same *form* for all observers O, but its value differs from observer to observer. Thus (37) represents a *private* space.

43. A public hyperbolic space. There is, however, a far more important difference between (35) and (37) than the relative simplicity of (35). Metric (35) can be shown to represent a *public* space, i.e. this dE^2 has the same *value* for all observers O, as well as the same form. To see this, we will first recover (35) from a different point of view.

44. Let us attempt to find a clock regraduation such that the new epoch coordinate T of an event takes the same value for all observers in the equivalence. In particular, this epoch coordinate must be the value assigned by the observer O' at the event itself. Hence, using the transformation formulae from O to O', since O' assigns a distance coordinate zero to the event, and since the epoch coordinates T and T'' used by O and O' are to be equal, we have

$$T - R/c = \theta_{21}(T'-0) = \theta_{21}(T), \tag{38}$$

$$T + R/c = \theta_{12}(T''+0) = \theta_{12}(T). \tag{39}$$

Hence
$$T = \tfrac{1}{2}\{\theta_{21}(T)+\theta_{12}(T)\}. \tag{40}$$

But after regraduation the generating function Ψ of the equivalence may be taken to be $\chi^{-1}\psi$, where $t = \chi(T)$ is the regraduation formula.

Hence
$$T = \tfrac{1}{2}\{\chi^{-1}\psi\alpha_{12}\psi^{-1}\chi(T)+\chi^{-1}\psi\alpha_{12}^{-1}\psi^{-1}\chi(T)\}. \tag{41}$$

This must not only be an identity in T; it must hold good for all observers O, i.e. for all values of the parameter α_{12}. Hence the right-hand side of (41) must be independent of α_{12}. Hence, to make the term in α_{12}^{-1} cancel the term in α_{12}, $\chi^{-1}\psi$ must be a linear function of a logarithm. We take then

$$\chi^{-1}\psi(T) = t_0\log(T/t_0)+t_0,$$

so that
$$T = t_0\log\frac{\psi^{-1}\chi(T)}{t_0}+t_0, \tag{42}$$

or
$$\chi(T) = \psi(t_0\,e^{(T-t_0)/t_0}).$$

The regraduation $t = \chi(T)$ is now known. The new signal functions $\theta_{12}(T)$ are given by

$$\theta_{12}(T) = \chi^{-1}\psi\alpha_{12}\psi^{-1}\chi(T)$$
$$= t_0\log\frac{\alpha_{12}\,t_0\,e^{(T-t_0)/t_0}}{t_0}+t_0$$
$$= T + t_0\log\alpha_{12}. \tag{43}$$

Relations (38) and (39) then give for the new distance R

$$R = \tfrac{1}{2}c\{\theta_{12}(T)-\theta_{21}(T)\} = ct_0\log\alpha_{12} = \text{const.}$$

We thus recover the relatively stationary equivalence. That is to say, the relatively stationary equivalence is the only one which gives rise to an absolute simultaneity, such that the various observers, members of the equivalence, assign the same epoch to any given event.

45. The metric of the relatively stationary equivalence, corresponding to the observer O's choice of a private Euclidean space in the uniform motion equivalence, has been obtained above, (35). Now the value of ds^2, for the case $\psi(t) \equiv t$, reduces, by (24), to

$$ds^2 = dt^2 - de^2/c^2 = dt^2 - \frac{dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)}{c^2}. \tag{44}$$

But by (31) and (32),

$$dt^2 - dr^2/c^2 = e^{2(T-t_0)/t_0}(dT^2 - dR^2/c^2).$$

Moreover, r is given by (34). Hence, by actual change of coordinates, (44) gives

$$ds^2 = e^{2(T-t_0)/t_0}\{dT^2 - dR^2/c^2 - t_0^2 \sinh^2(R/ct_0)(d\theta^2 + \sin^2\theta \, d\phi^2)\}. \tag{45}$$

In this relation ds^2 is the same for all observers, members of the equivalence. But the coordinate T now has the same property. Hence the value of the longer bracket in (45) is the same for all the observers. But dT is also the same for all the observers. Hence

$$dR^2 + (ct_0)^2\sinh^2(R/ct_0)(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{46}$$

is the same for all the observers, members of the now relatively stationary equivalence. But this is just dE^2, the spatial metric of the equivalence, as given by (35). Hence dE^2 represents a *public* space.

In other words, if we choose for each observer O in the uniform motion equivalence a private Euclidean space

$$de^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

and then regraduate each observer's clock so that the equivalence becomes relatively stationary, then the corresponding spatial measures dE^2 of each observer by his regraduated clock are all equal and of the same form. Moreover dE^2 is independent of the epoch coordinate. Lastly, the epoch coordinate of any given event is the same for each observer. There is thus a public time and a public (hyperbolic) space.

46. Ambiguity of contemporary physics. We shall show in due course that this public time and public space are the actual time and space used in classical dynamics, whereas the time and private spaces of the uniform motion equivalence are those used in optics and in Maxwell's equations. Contemporary physics thus has an ambiguity running through it, inasmuch as it confuses the time-variables used in two distinct domains of investigation.

THE SUBSTRATUM

47. Recapitulation of the notion of an equivalence. An equivalence, as defined in the two preceding chapters, is a class of types of motion. The various members A, B, C, etc., who satisfy the conditions $A \equiv B \equiv C \equiv ...$, are such that if any one member considers himself as 'at rest', the relative motions of all the others have 'something in common'. This 'something' is the generating function ψ of the equivalence. We have seen that the equivalence generated by $\psi(t) \equiv t$, for example, consists of particles in uniform relative motion, which have separated from a common point at a common epoch. We have seen also that any other equivalence, generated by some specific function ψ, can be transformed into the uniform motion equivalence by suitable clock-regraduation, and that accordingly there is only *one* equivalence. Moreover, an equivalence comprises a set of particle-observers whose clocks, in a well-defined sense, can be described as *congruent* to one another, and which therefore possess a common system of time-keeping. But so far we have not imposed any density-distribution on the particles, members of the equivalence.

48. The free particle. Our next object is to identify one of the possible modes of clock-graduation of an equivalence with the 'uniform time' of Newtonian physics. This 'uniform time' has the property that a 'free particle' in 'empty space' is supposed to move uniformly relative to an 'inertial' frame. But Newtonian physics is silent as to what constitutes an inertial frame. An inertial frame in practice is one that can be regarded as at local rest, and therefore 'unaccelerated'. It is natural to regard the members of an equivalence as defining standards of local rest, and we shall pursue the consequences of this view.

The concept of 'a free particle in empty space' is, however, beset with difficulties. Prima facie it implies that the particle in question is not subject to any gravitational field, and therefore that it is at a large distance from any attracting matter. But the attracting matter of the universe is aggregated into galaxies, which extend in unending number through space; and it is inconceivable that they should ever possess a boundary. It is therefore impossible to consider a particle at a great distance from all attracting matter when

by 'great distance' we mean an indefinitely great distance. It is therefore necessary to replace the concept of a 'free particle in empty space' by the concept of a free particle in the presence of the universe at large.

49. The homogeneous equivalence. In our abstract scheme, having settled our modes of time-keeping, we shall have to consider the motion of a free particle in the presence of an equivalence. But if our analysis is to represent the effects of gravitation, it is clearly necessary to impose a density-distribution on the equivalence. The most natural density-distribution to select for investigation in the first instance is clearly a *homogeneous* distribution, if such can be defined. But a little investigation shows that a crude definition of homogeneity in terms of equality of number of particles per unit volume in the reckoning of any observer will not suffice. For a distribution homogeneous in the experience of one observer will not necessarily be homogeneous in the experience of another observer. We cannot regard as satisfactory a definition of a homogeneous equivalence as one such that N, the number of particles per unit volume at epoch t in the experience of a given observer O, is everywhere the same; for to another observer the various contents of elementary volumes will be counted at *different* times t', since in general another observer O' will not regard as simultaneous, events which are simultaneous to O. We must therefore generalize the notion of homogeneity. We do this by imposing the condition that $N(r,t)$, the number of particles at distance r at epoch t per unit volume in O's measures shall be the same function of r and t to all other observers, members of the equivalence. That is to say, that if the density distribution is $N(r,t)$ to O, and the same density-distribution is $N'(r',t')$ to O', then $N \equiv N'$ when (r,t), (r',t') are respectively coordinates used by O and by O' for the same event. If it is possible to obtain such a distribution, we shall call it the *homogeneous equivalence*; and because of its fundamental importance in dynamics, we shall call the homogeneous equivalence a *substratum*.

50. Density distribution in hyperbolic space. Clearly when an absolute simultaneity amongst the members of an equivalence exists, the above definition of homogeneity must reduce to the ordinary one: the density must not only be the same function of his coordinates to every observer, but must take the same value everywhere.

It follows that for the relatively stationary equivalence, the corresponding substratum density must be constant. The spatial metric giving a public space for the relatively stationary substratum we saw to be

$$dE^2 = dR^2 + c^2 t_0^2 \sinh^2(R/ct_0)(d\theta^2 + \sin^2\theta \, d\phi^2).$$

The volume element in this space is

$$(ct_0)^2 \sinh^2(R/ct_0)\sin\theta \, d\theta d\phi dR,$$

and consequently the density-distribution for the relatively stationary substratum must be of the form

$$N(ct_0)^2 \sinh^2(R/ct_0)\sin\theta \, d\theta d\phi dR,$$

where N is a constant independent of T.

51. The substratum in t-measure. The question arises, What is the corresponding description of the substratum in uniform relative motion, when t-measure is employed? For the substratum in uniform relative motion, the transformation formulae are of Lorentz type, and the density-distribution can be found directly as follows.

52. Let O, O' be two observers, members of the equivalence in uniform relative motion. Let u, v, w be the velocity-components of any member of the equivalence to O, u', v', w' the velocity-components of the same member to O'. Then by the velocity-transformation formulae, due originally to Einstein, if $(U, 0, 0)$ is the relative velocity of O' with respect to O, we have

$$u' = \frac{u-U}{1-uU/c^2}, \qquad v' = \frac{v(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}, \qquad w' = \frac{w(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}.$$

These velocity-transformation formulae follow at once by differentiation of the Lorentz formulae connecting O and O' (which we have obtained independently in §§ 24, 30), namely

$$x' = \frac{x-Ut]}{(1-U^2/c^2)^{\frac{1}{2}}}, \qquad t' = \frac{t-Ux/c^2}{(1-U^2/c^2)^{\frac{1}{2}}}, \qquad y = y', \qquad z = z',$$

with

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt}, \qquad w = \frac{dz}{dt},$$

$$u' = \frac{dx'}{dt'}, \qquad v' = \frac{dy'}{dt'}, \qquad w' = \frac{dz'}{dt'}.$$

Let now $f_O(u, v, w)\,dudvdw$ be the number of particles in the homogeneous equivalence (or substratum) to O, with velocities lying

between u and $u+du$, v and $v+dv$, w and $w+dw$. When O' counts the same particles, let him find $f_{O'}(u', v', w')\,du'dv'dw'$. It is to be noticed that since we are discussing the uniform relative motion equivalence, there is no need to include mention, in f, of the epoch t; for the velocities are constant. Then

$$f_O(u, v, w)\,dudvdw = f_{O'}(u', v', w')\,du'dv'dw'.$$

But if the distribution is to be homogeneous, in the sense defined above, we must have also

$$f_O \equiv f_{O'} \equiv f,$$

say. Hence f must satisfy the functional equation

$$f(u, v, w) = f(u', v', w')\,\frac{\partial(u', v', w')}{\partial(u, v, w)}.$$

It is readily found that

$$\frac{\partial(u', v', w')}{\partial(u, v, w)} = \frac{(1-U^2/c^2)^2}{(1-uU/c^2)^4}.$$

Hence f must satisfy

$$f(u, v, w) \equiv f\!\left(\frac{u-U}{1-uU/c^2}, \frac{v(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}, \frac{w(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}\right)\frac{(1-U^2/c^2)^2}{(1-uU/c^2)^4},$$

for all values of $|U| < c$, and two similar functional equations with $(U, 0, 0)$ replaced by $(0, U, 0)$ and $(0, 0, U)$. It can be shown† that the most general solution of these functional equations is

$$f(u, v, w)\,dudvdw = \frac{B\,dudvdw}{c^3\{1-(u^2+v^2+w^2)/c^2\}^2}, \tag{1}$$

where B is an arbitrary constant. This is therefore the velocity-distribution in the homogeneous equivalence in uniform relative motion.

53. Spatial distribution. To find the spatio-temporal distribution corresponding to this, we notice that for sufficiently large values of t, we must have

$$u \sim \frac{x}{t}, \qquad v \sim \frac{y}{t}, \qquad w \sim \frac{z}{t}.$$

For, the motion being uniform, we must have

$$x = ut + \text{const., etc.}$$

† A detailed proof was given in *W. S.* (1935).

Inserting this in (1), we get that the number of particles inside $dx\,dy\,dz$ is

$$\frac{B\dfrac{dx}{t}\dfrac{dy}{t}\dfrac{dz}{t}}{c^3\left(1-\dfrac{\sum x^2}{c^2t^2}\right)^2}.$$

This must be $N\,dx\,dy\,dz$, where N is the particle-density at distance r at time t, as reckoned by O. The number of particles, in O's reckoning, at (x,y,z) at epoch t is accordingly, when t is sufficiently large,

$$N\,dx\,dy\,dz = \frac{Bt\,dx\,dy\,dz}{c^3(t^2-r^2/c^2)^2}. \tag{2}$$

The density N at time t in O's reckoning will be found to satisfy exactly the hydrodynamical equation of continuity

$$\frac{\partial N}{\partial t}+\sum\frac{\partial}{\partial x}(Nu)=0, \tag{3}$$

with $u=x/t, \qquad v=y/t, \qquad w=z/t.$ (4)

Moreover, the motion (4) makes all the members of the substratum coincide at $t=0$, in accordance with the general property of an equivalence. Hence if $\mathbf{P}$ is the vector position of a typical member of the equivalence at epoch t, $N(\mathbf{P},t)\,dx\,dy\,dz$ the density-distribution of the substratum at P at epoch t, then

$$N(\mathbf{P},t)=\frac{Bt}{c^3(t^2-\mathbf{P}^2/c^2)^2}, \tag{5}$$

whilst the velocity $\mathbf{V}$ of any member is given by

$$\mathbf{V}=\mathbf{P}/t. \tag{6}$$

Relations (5) and (6) will now be found to constitute an exact solution of the problem of obtaining a homogeneous uniformly-moving equivalence. It may readily be verified, in fact, that if from (5) and (6) we calculate the particle-density distribution to another observer O' at epoch t' at position vector $\mathbf{P}'$, then we find

$$\frac{Bt'\,dx'\,dy'\,dz'}{c^3(t'^2-\mathbf{P}'^2/c^2)^2}. \tag{7}$$

54. Properties of the substratum. The substratum or homogeneous equivalence in uniform relative motion has many remarkable properties. In the first place it is not homogeneous in the ordinary sense of homogeneity in the instantaneous present of any observer O.

Instead, it is distributed with spherical symmetry round O, with particle-density increasing from B/c^3t^3 at O to infinity at distance ct. O is the centre of the distribution, in his own view, but if he moves his headquarters to any other fundamental particle O', a member of the substratum, then it again appears to be distributed with spherical symmetry round O', and with density increasing outwards from O'. To any observer, a member of the system, the system appears to occupy the interior of an expanding sphere of radius ct. There are no particles at the actual boundary, for such particles would be moving with the speed of light. The set of particles therefore forms an *open* set of points. There is a natural origin of time, $t = 0$, at which the system appears to have come into existence, at the origin O. But this origin may equally be taken to be at any other member of the system. We can therefore call $t = 0$ the 'epoch of creation' of the system. Every particle is in radially outward motion relatively to every other, with a speed proportional to its distance. No meaning attaches to the questions 'What was, before creation?' or 'What is, outside the expanding sphere $r = ct$?' The system appears to create the space it needs, as it expands. To an observer inside the system, the system has all the properties of infinite space, for it is impossible to assume a velocity which will take the observer outside the system. It is therefore illegitimate to inquire what the system would look like from the outside; the only legitimate observers are observers inside the system.

A distant, receding member, will assign to an event at himself an epoch t', where, if t is the epoch assigned by the observer at home, we have $t' = t(1 - V^2/c^2)^{\frac{1}{2}}$. The particles near the confines of the expanding system, for which V is nearly c, will therefore have very early local time. Hence if the particles are conceived as having an evolutionary history, the particles near $r = ct$ will appear to be in a very early stage of that history, very little removed, in fact, from the epoch of creation. The phenomenon of the creation of the system will appear to have only just taken place for particles near $r = ct$, and the singularity in density at the origin at time $t = 0$ has its counterpart in the singularity in density at the distance $r = ct$ at epoch t in the experience of O. There are thus an infinite number of members of the system which in the view of O appear to have only just been created. But it may be shown that whatever velocity O acquires, he can never overtake events with local epochs earlier

than the epoch of the event of his leaving home. Thus the flow of time to any observer is irreversible, and the whole system is in fact irreversible.

55. Regraduation to give a relatively stationary substratum.

Such is the description of the substratum or homogeneous equivalence to an observer using t-measure of time, that is to say, to any observer (a member of the equivalence) whose clock is so graduated that the rate of recession of any other member appears uniform. Let us consider its description to the same members when they have regraduated their clocks so as to convert the uniform motion into stationariness.

We know that the desired regraduation of clocks is given by

$$T = t_0 \log(t/t_0) + t_0,$$

where T is the label of the instant labelled t previously. The corresponding transformation of coordinates we have seen to be given by

$$T - R/c = t_0 \log \frac{t - r/c}{t_0} + t_0, \tag{8}$$

$$T + R/c = t_0 \log \frac{t + r/c}{t_0} + t_0. \tag{9}$$

These yield

$$t = t_0 e^{(T-t_0)/t_0} \cosh(R/ct_0), \tag{10}$$

$$r = ct_0 e^{(T-t_0)/t_0} \sinh(R/ct_0), \tag{11}$$

so that $\qquad t^2 - r^2/c^2 = t_0^2 e^{2(T-t_0)/t_0}. \tag{12}$

When the observer adopts a private Euclidean space de^2 in t-measure, he takes the number of particles in the elementary volume $r^2 dr d\omega$ (where $d\omega = \sin\theta \, d\theta d\phi$) to be

$$\frac{B t r^2 dr d\omega}{c^3 (t^2 - r^2/c^2)^2}. \tag{13}$$

This number of particles is to be found in the element of volume $r^2 dr d\omega$ at the given epoch t. The corresponding differentials dR and dT are therefore given, by (8) and (9), by

$$dT - dR/c = -\frac{t_0 \, dr/c}{t - r/c}, \tag{14}$$

$$dT + dR/c = \frac{t_0 \, dr/c}{t + r/c}, \tag{15}$$

where we have put $dt = 0$. The particle at $(r + dr, t)$ is now found

at $(R+dR, T+dT)$, but since the particles are now regarded as relatively stationary, the same particle will also be found at $(R+dR, T)$. Hence if we solve (14) and (15) for dR, this differential will include all the particles previously included in dr at epoch t. We get

$$dR = \frac{t_0\,t\,dr}{t^2 - r^2/c^2},$$

or

$$dr = dR e^{(T-t_0)/t_0} \mathrm{sech}(R/ct_0). \tag{16}$$

Now let ν be the spatial density in the space of metric dE^2 which corresponds to T-measure. Then since the number of particles in corresponding elementary volumes is independent of the space adopted, we have, by (13), on using (10), (11), and (16),

$$\nu(ct_0)^2 \sinh^2(R/ct_0)\,dR\,d\omega$$

$$= \frac{B(t_0\,e^{(T-t_0)/t_0}\cosh R/ct_0)(ct_0\,e^{(T-t_0)/t_0}\sinh R/ct_0)^2 e^{(T-t_0)/t_0}\,\mathrm{sech}\,R/ct_0\,dR\,d\omega}{c^3(t_0^2\,e^{2(T-t_0)/t_0})^2},$$

which gives

$$\nu = B/c^3 t_0^3. \tag{17}$$

This is the value denoted previously by N (§ 50) and, as there anticipated, in the space dE^2 the density is constant, independent of T and R. Moreover, it is equal to the density at the origin in t-measure, at epoch $t = t_0$.

56. Description of the relatively stationary substratum.

In t-measure (uniform relative motion) the substratum is confined to the interior of the expanding sphere $r = ct$, as we have seen. Now the position $r = ct$ corresponds in T-measure (relatively stationary equivalence) to $R = \infty$. Thus the effect of regraduating observers' clocks from t to T is to project the interior of the sphere $r = ct$ in O's private Euclidean space into the whole of the public hyperbolic space dE^2, which extends to infinity. Instead of a singularity in density at $r = ct$, we have now an everywhere uniform density, but the total number of particles is infinite as before. Instead of being concentrated towards the boundary $r = ct$, they are spread uniformly through infinite space. Instead of a density increasing outwards, and everywhere decreasing as time advances, we have a density stationary in time and constant in space. Instead of uniform outward motion we have a state of relative rest. Instead of the epoch of any event depending on the observer describing the event, we have a world-wide simultaneity.

57. Let V_0 be the (constant) recession velocity of a member of the substratum in t-measure. Then $V_0 = r/t$. Hence if R_0 is the corresponding (constant) distance coordinate in T-measure, we have by (10) and (11),
$$V_0 = c \tanh R_0/ct_0. \tag{18}$$

This verifies at once that a member of the substratum in t-measure has a constant distance coordinate in T-measure. Moreover, as $V_0 \to c$, $R_0 \to \infty$.

58. Fundamental particles and fundamental observers. We have called an equivalence in which the members are 'homogeneously' distributed a *substratum*. We shall call the members of the substratum *fundamental particles*, and the observers associated with them *fundamental observers*. Every fundamental observer is equivalent to every other fundamental observer, and the clocks of all fundamental observers are congruent to one another. Every fundamental particle is equally the centre of all the others, and every fundamental observer sees the other fundamental particles arranged with spherical symmetry round himself in t-measure, or arranged uniformly in T-measure. To the observer O, there is, in t-measure, at every point r of the interior of the sphere $r = ct$ a characteristic velocity V given by $V = r/t$, in a direction outward from O. This velocity is proportional to the distance from O, the coefficient of proportionality being the reciprocal of the age of the system reckoned from the natural origin of time.

It is clear from our construction of an equivalence that particles with a motion other than the characteristic motion at any point are *not* equivalent to fundamental particles, and that they cannot be provided with clocks congruent to the clocks carried by the totality of fundamental observers. Fundamental observers thus constitute the totality of observers having a common system of time-keeping, and thus they are the only observers whose descriptions of phenomena inside the substratum can be expected to coincide. The motion of the fundamental observers (in t-measure), or their state of rest (in T-measure) constitute the natural frames of reference for the description of phenomena; and it is only to such observers that the laws governing such phenomena can be expected to be identical.

For example, not *all* frames in uniform relative motion are equivalent, but only those which separated from the remainder at

the singular event, $t = 0$. A frame in uniform motion relative to a fundamental particle will not in general be equivalent to the frames associated with fundamental particles. The principle of relativity of uniform motion is thus confined to a much smaller class of frames of reference than in current physics, when account is taken of congruent time-keeping.

59. Application to the galaxies. The application of the theory of the equivalence and the substratum is to the system of the external galaxies. The displacements of their spectral lines to the red, interpreted according to the rules of the Doppler effect, indicate that they are all receding from our own galaxy and from one another with speeds proportional to their distances. We shall examine later the behaviour of photons, and their frequencies, in the substratum, and the interpretation of the red-shift when T-measure is used. But provisionally we can identify the external galaxies with fundamental particles. Each nebular nucleus then determines a state of local rest, and a system of time-keeping congruent with our own. The laws governing phenomena, described by observers located at the nebular nuclei, may be expected to be the same as the laws governing phenomena in our own galaxy, viewed from the nucleus of our galaxy. But frames in motion relative to a nebular nucleus will not in general be equivalent to the nebular nucleus concerned, or to ourselves, even if the motion be relatively uniform.

If this identification is justifiable, it would appear that the extra-galactic nebulae separated from one another at a time ago given by the ratio of the distance to the recession velocity as about 2×10^9 years. This in t-measure would be the 'date of creation'. But an infinity of other systems of time-keeping are equally legitimate, and in particular, in T-measure the 'date of creation' would be 'minus infinity'. It therefore becomes important to attempt to identify the time-scales used in describing the different kinds of physical phenomena with the time-scales of our abstract theory. It must be remembered that choice of time-scale never affects phenomena, but only the description of phenomena. For example, since in the t-description of the substratum the local time for events near the frontier $r = ct$ is very early, i.e. near $t = 0$, so the evolutionary stage of members of the substratum at great distances in T-measure will also be very early, in spite of the fact that the substratum is now static.

60. Transition from kinematics to dynamics. So far we have considered an equivalence or a substratum as a *given* set of particles in motion. We have not examined whether, if a given state of motion exists at epoch t and there is a kinematically consequent state of motion at epoch $t+dt$, then the one state will pass into the other state. For example, will the state of uniform motion in the t-description of the substratum in O's private space continue of itself? To answer these questions is to pass from kinematics to dynamics. We therefore address ourselves in the next chapter to the fundamental problem of dynamics, which is to ascertain the motion of a free test-particle in the presence of the substratum. We shall examine this question first in t-measure.

DYNAMICS

V

THE MOTION OF A FREE TEST-PARTICLE

61. Use of the Lorentz formulae. Our object is to obtain the equation of motion of a free test-particle in motion in any manner in the presence of the substratum.

Let the substratum be described in t-measure. That is to say, let the time-scale be such that the fundamental particles are in uniform relative motion. Take one fundamental particle O as origin. Then the velocity vector $\mathbf{V_0}$ of the fundamental particle at position vector $\mathbf{P}$ with respect to O is given by

$$\mathbf{V_0} = \mathbf{P}/t, \tag{1}$$

at epoch t. Since the fundamental particles form an equivalence in uniform relative motion, the coordinates assigned by any two fundamental observers to a given event will be connected by the *Lorentz formulae*, Chap. II, equations (38) and (39). It is convenient to establish first some consequences of these formulae.

62. Properties of the Lorentz formulae. Take temporarily two observers in relative motion along the x-axis separating with speed U. Then if they respectively assign coordinates (x, y, z, t), (x', y', z', t') to a given event, the Lorentz formulae give

$$x' = \frac{x - Ut}{(1 - U^2/c^2)^{\frac{1}{2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - Ux/c^2}{(1 - U^2/c^2)^{\frac{1}{2}}}, \tag{2}$$

where we have chosen the positive direction of the x-axis as from the observer O using unprimed coordinates towards the observer O' using primed coordinates. We verify immediately that

$$t^2 - \frac{x^2 + y^2 + z^2}{c^2} = t'^2 - \frac{x'^2 + y'^2 + z'^2}{c^2}. \tag{3}$$

The same result follows whatever is the direction of relative motion of O and O'. Consequently for any event $(t, \mathbf{P})$, $(t', \mathbf{P}')$, we have

$$t^2 - \mathbf{P}^2/c^2 = t'^2 - \mathbf{P}'^2/c^2. \tag{4}$$

We shall call the common value of these expressions X. The quantity X is an invariant, of the dimensions of the square of a time.

63. If the same observers O and O' assign velocities (u, v, w), (u', v', w') to a particle, we have seen that

$$u' = \frac{u-U}{1-uU/c^2}, \qquad v' = \frac{v(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}, \qquad w' = \frac{w(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2}. \quad (5)$$

Hence

$$1 - \frac{u'^2+v'^2+w'^2}{c^2} = \left(1 - \frac{u^2+v^2+w^2}{c^2}\right)\left(1 - \frac{U^2}{c^2}\right)\Big/(1-uU/c^2)^2. \quad (5')$$

We shall put

$$Y = 1 - \frac{u^2+v^2+w^2}{c^2} = 1 - \frac{V^2}{c^2}, \qquad\qquad (5'')$$

$$Y' = 1 - \frac{u'^2+v'^2+w'^2}{c^2} = 1 - \frac{V'^2}{c^2}. \qquad\qquad (5''')$$

It is evident that Y and Y' are covariants but not invariants.

Now consider the expression

$$dt^2 - \frac{dx^2+dy^2+dz^2}{c^2},$$

where $(t+dt, x+dx, y+dy, z+dz)$, (t, x, y, z) are two neighbouring events. We readily find

$$dt^2 - \frac{dx^2+dy^2+dz^2}{c^2} = dt'^2 - \frac{dx'^2+dy'^2+dz'^2}{c^2}.$$

If now the two events occur on the path of a particle moving with velocity V, then $dx = u\,dt$, $dy = v\,dt$, $dz = w\,dt$, and we have

$$Y^{\frac{1}{2}}dt = Y'^{\frac{1}{2}}dt', \qquad\qquad (6)$$

so that $Y^{\frac{1}{2}}dt$ is an invariant. It follows that

$$\left(\frac{dx}{Y^{\frac{1}{2}}dt},\ \frac{dy}{Y^{\frac{1}{2}}dt},\ \frac{dz}{Y^{\frac{1}{2}}dt},\ \frac{c\,dt}{Y^{\frac{1}{2}}dt}\right)$$

obeys the same laws of transformation in passing from O to O' as

$$(x, y, z, ct).$$

That is to say, $\qquad\qquad \left(\dfrac{u}{Y^{\frac{1}{2}}},\ \dfrac{v}{Y^{\frac{1}{2}}},\ \dfrac{w}{Y^{\frac{1}{2}}},\ \dfrac{c}{Y^{\frac{1}{2}}}\right)$

also obeys the Lorentz transformation. We shall call such assemblies '4-vectors'. Thus $(\mathbf{P}, ct)$ and $(\mathbf{V}/Y^{\frac{1}{2}}, c/Y^{\frac{1}{2}})$ are 4-vectors; we call $\mathbf{P}$, and $\mathbf{V}/Y^{\frac{1}{2}}$ their 'space-components', ct and $c/Y^{\frac{1}{2}}$ their 'time-components'. (The coefficient c is used to make all components of a given 4-vector of the same physical dimensions.)

64. Now consider dX/dt. We have

$$\frac{dX}{dt} = 2\left(t - \frac{\mathbf{P} \cdot \mathbf{V}}{c^2}\right) = 2Z, \tag{7}$$

say. Hence $Z/Y^{\frac{1}{2}}$ is an invariant, since X and $Y^{\frac{1}{2}} dt$ are invariants. The invariant $Z/Y^{\frac{1}{2}}$ is of the dimensions of time. It follows that Z is a covariant. It follows also that

$$Z/X^{\frac{1}{2}}Y^{\frac{1}{2}}$$

is an invariant, of zero physical dimensions. We shall call it $\xi^{\frac{1}{2}}$.

65. It is convenient to summarize this notation, which will be adhered to throughout this book: $(\mathbf{P}, t)$ is an event as described by a fundamental observer O, V is the velocity, to O, of any particle passing through the position P at epoch t. Then

$$V = d\mathbf{P}/dt, \tag{8}$$

$$X = t^2 - \mathbf{P}^2/c^2, \qquad Y = 1 - \mathbf{V}^2/c^2, \qquad Z = t - \mathbf{P} \cdot \mathbf{V}/c^2, \tag{9}$$

$$\xi = Z^2/XY. \tag{10}$$

The reason for this notation is to emphasize that X, Y, Z, ξ are not constants, but invariants or covariants. The use of the second member Y of the triplet X, Y, Z, to denote $1 - \mathbf{V}^2/c^2$ recalls the use in 'special relativity' of β (the second member of the triplet α, β, γ) to denote $(1 - U^2/c^2)^{\frac{1}{2}}$, where U is an observer's velocity relative to some other observer. But in the present notation, V always denotes the velocity of a particle. Since X, Y, Z, ξ are respectively of dimensions 2, 0, 1, 0 in t, the physical dimensions of any proposed combination of them is apparent at sight.

66. Problem of the free test-particle. We can now investigate the problem of the equation of motion of a free test-particle in the substratum. Consider a particle (not a fundamental particle) passing through the position P at epoch t with velocity V, all as measured by an observer O at the origin. Then this free test-particle will have some particular acceleration $d\mathbf{V}/dt$. This acceleration may depend on P, t, V as arguments; and it may involve also the conventional constant c and the coefficient B defining the density in the substratum. The variables P, t, V must be considered as capable of independent variation; for through any arbitrary position P, at an arbitrary epoch t, we can suppose a free particle, passing, or projected, with an arbitrary velocity V. The complete trajectory of

the free test-particle should be obtainable by integration of the differential equation obtained by determining dV/dt, the instantaneous acceleration as measured by O, as a function of P, t, and V. The question arises whether this function can be found without appeal to any empirical physical laws. If it can, we shall have bridged the gap between kinematics and dynamics, a gap hitherto as definite as that between the mineral and vegetable kingdoms.

67. The question whether this problem is soluble without physical appeal is a question in pure logic: if a free test-particle is moving through the position P at instant t with velocity V (all with reference to a fundamental observer O as origin) in the presence of the substratum, is it possible to infer its acceleration dV/dt purely from the definition and properties of the substratum? That is, can we establish a *dynamical theorem* from the properties of the substratum defined purely kinematically?

68. **The 4-vector form of the equations of motion of a free test-particle.** To investigate this, let us consider instead of the 3-vector dV/dt, the corresponding 4-vector

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right), \qquad \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right). \tag{11}$$

We wish to express this as a function of P, t, and V. Now this 4-vector can be resolved along the only 4-vectors at our disposal, namely

$$(P, ct) \quad \text{and} \quad \left(\frac{V}{Y^{\frac{1}{2}}}, \frac{c}{Y^{\frac{1}{2}}}\right). \tag{12}$$

And when it is so resolved, the coefficients must be 4-scalars, for if not, an inconsistency would arise when the observing headquarters O was transformed to another fundamental observer O'. Moreover, the coefficients must be such that (11), when expressed as a linear function of the 4-vectors (12), is of dimensions *one* in length and *minus two* in time. (We can of course alternatively consider all our variables as having only time-dimensions, but we then need to ensure that they continue to be consistent with one another under change of c; this comes to the same thing; factors of the type ct are conveniently described as having the dimensions of a length.)

Now the only invariant at our disposal of dimensions minus two in time is $1/X$; hence the only way in which the 4-vector (P, ct) can appear in the expression for (11) is with α/X as coefficient, where α

is a 4-scalar or invariant of zero dimensions. The only invariants at our disposal of dimensions minus one in time are $Y^{\frac{1}{2}}/Z$ and $X^{-\frac{1}{2}}$; it is immaterial in the sequel which of these we take; hence the 4-vector $(V/Y^{\frac{1}{2}}, c/Y^{\frac{1}{2}})$ can only appear with $\beta Y^{\frac{1}{2}}/Z$ as a coefficient, where β is again a 4-scalar or invariant of zero dimensions. Hence the expression for (11) as a function of P, t, V must be of the form

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right) = \alpha\frac{P}{X} + \beta\frac{Y^{\frac{1}{2}}}{Z}\frac{V}{Y^{\frac{1}{2}}}, \tag{13}$$

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right) = \alpha\frac{ct}{X} + \beta\frac{Y^{\frac{1}{2}}}{Z}\frac{c}{Y^{\frac{1}{2}}}. \tag{14}$$

Differentiating out the first left-hand side we get

$$\frac{1}{Y} \frac{dV}{dt} + \frac{V}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(\frac{1}{Y^{\frac{1}{2}}}\right) = \alpha\frac{P}{X} + \beta\frac{Y^{\frac{1}{2}}}{Z}\frac{V}{Y^{\frac{1}{2}}}. \tag{15}$$

Multiply this last scalarly by V/c^2. Then, since

$$\frac{d}{dt}\left(\frac{1}{Y^{\frac{1}{2}}}\right) = \frac{1}{Y^{\frac{1}{2}}}\left(\frac{V}{c^2}.\frac{dV}{dt}\right), \tag{16}$$

(15) yields $$\left(\frac{1}{Y} + \frac{1-Y}{Y^2}\right)\frac{V}{c^2}.\frac{dV}{dt} = \frac{\alpha}{X}\frac{P.V}{c^2} + \frac{\beta}{Z}\frac{V^2}{c^2},$$

or $$\frac{1}{Y^2}\frac{V}{c^2}.\frac{dV}{dt} = \frac{\alpha}{X}(t-Z) + \frac{\beta}{Z}(1-Y). \tag{17}$$

But (14) gives $$\frac{1}{Y^2}\frac{V}{c^2}\frac{dV}{dt} = \frac{\alpha t}{X} + \frac{\beta}{Z}. \tag{18}$$

Comparing (17) and (18), we see that we must have

$$\alpha\frac{Z}{X} + \beta\frac{Y}{Z} = 0,$$

or $$\beta = -\alpha\frac{Z^2}{XY}. \tag{19}$$

Hence the 4-vector form of the equations of motion, (13) and (14), becomes

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right) = \frac{\alpha}{X}\left(P - V\frac{Z}{Y}\right), \tag{20}$$

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right) = \frac{\alpha}{X}\left(ct - c\frac{Z}{Y}\right). \tag{21}$$

The coefficient α is a 4-scalar of dimensions zero. The only 4-scalar of dimensions zero at our disposal is ξ, or Z^2/XY. Hence α must be a function of ξ, say $$\alpha = G(\xi). \tag{22}$$

Hence the equations of motion (20) and (21) take the form

$$\frac{1}{Y^4}\frac{d}{dt}\left(\frac{V}{Y^4}\right) = \frac{G(\xi)}{X}\left(P - V\frac{Z}{Y}\right),\tag{23}$$

$$\frac{1}{Y^4}\frac{d}{dt}\left(\frac{c}{Y^4}\right) = \frac{G(\xi)}{X}\left(ct - c\frac{Z}{Y}\right).\tag{24}$$

69. Three-vector form of the equation of motion. These two relations amount to four scalar relations of which the fourth (24) can be deduced from the first three represented by (23). In other words, the time-component relation (24) is a consequence of the 3-vector space-component relation (23). This is readily seen by multiplying (23) scalarly by V/c^2, when it gives

$$\frac{1}{Y}\frac{V}{c^2}\cdot\frac{dV}{dt} + \frac{1-Y}{Y^2}\frac{V}{c^2}\cdot\frac{dV}{dt} = \frac{G(\xi)}{X}\left\{(t-Z) - \frac{Z}{Y}(1-Y)\right\},$$

or

$$\frac{1}{Y^2}\frac{V}{c^2}\cdot\frac{dV}{dt} = \frac{G(\xi)}{X}\left(t - \frac{Z}{Y}\right),\tag{25}$$

which is just (24).

We can readily derive now an equation for dV/dt itself. For (23) gives

$$\frac{1}{Y}\frac{dV}{dt} + \frac{V}{Y^2}\left(\frac{V}{c^2}\cdot\frac{dV}{dt}\right) = \frac{G(\xi)}{X}\left(P - V\frac{Z}{Y}\right),$$

substituting in this for $\dfrac{V}{c^2}\cdot\dfrac{dV}{dt}$ from (25) we get

$$\frac{1}{Y}\frac{dV}{dt} = \frac{G(\xi)}{X}\left\{\left(P - V\frac{Z}{Y}\right) - V\left(t - \frac{Z}{Y}\right)\right\} = \frac{G(\xi)}{X}(P - Vt).$$

Hence

$$\frac{dV}{dt} = \frac{Y}{X}(P - Vt)G(\xi).\tag{26}$$

This is a pure 3-vector relation, but its form is invariant under a Lorentz transformation from O to any other fundamental observer O'. The same is true of (23) and (24), as is readily recognizable from their form. The time-component corresponding to (26) is the identity

$$\frac{dc}{dt} = \frac{Y}{X}(ct - ct)G(\xi),$$

c being of course a constant. Throughout our analysis we shall find two strands of equations, the Lorentz-invariant 4-vector equations of the type (23) and (24), in which the various terms are components of 4-vectors, and Lorentz-invariant 3-vector equations of the type (26), in which the terms are 3-vectors but not the space parts of 4-vectors.

70. Conversely, from the 3-vector equation (26), it is possible to deduce both (23) and (24). Scalar multiplication of (26) by $\mathbf{V}/c^2$ gives (24) at once, and then on constructing by direct differentiation the 4-vector component

$$\frac{1}{\gamma^{\frac{1}{4}}}\frac{d}{dt}\left(\frac{\mathbf{V}}{\gamma^{\frac{1}{4}}}\right),$$

and using (24) we recover (23).

71. The substratum as a dynamical system. From (26) a fundamental deduction can be made. The acceleration $d\mathbf{V}/dt$ vanishes if $\mathbf{V}$ is such that $\mathbf{P}-\mathbf{V}t = 0$, i.e. if $\mathbf{V} = \mathbf{P}/t$. That is to say, if a free test particle has the velocity of the fundamental particle in its neighbourhood, its acceleration is zero, and so it continues to accompany the same fundamental particle. But this is as much as to say that each fundamental particle behaves as a free particle. The fundamental particles were first *prescribed* to move with the velocities $\mathbf{V}_0 = \mathbf{P}/t$. We now see that they will maintain this motion of their own accord. In other words, the substratum, originally defined as a *kinematic* system, with *prescribed* motions, is also a *dynamical* system, which will continue in the prescribed motion by itself. From one point of view this is self-evident, since each fundamental observer regards the particle on which he is situated as central amongst the rest of the particles, and therefore without any tendency to be accelerated in any one particular direction. But it is reassuring to have an analytical proof.

72. The dimensional argument. A word may be said at this stage concerning relation (22), $\alpha = G(\xi)$. If the substratum contained in its description a scalar constant of the dimensions of a time, we could combine this with X to obtain another argument of dimensions zero in the time, and α might be of the form $\alpha = G(\xi, X)$. But the only constants occurring in the description of the substratum are c and B; of these c is of the dimensions of a velocity, and the value of B, which is dimensionless, must be irrelevant to the properties of the substratum, since it is entirely arbitrary what we choose to count as a particle. If the substratum had been constructed as a model of the universe, any combination of 'constants of nature' of the dimensions of a time might enter into the expression for α as a function of X. But we are considering the substratum simply as an abstract concept, not as a model of some physical system; and the motion

of a free test-particle in its presence must be capable of being stated in terms of the abstract properties of the conceptual substratum. Since no 'constants' occur in the definition of the substratum except the conventional c and B, only these can occur in the description of that property of the substratum which is represented by the equation of motion of a free test-particle. And thus α can depend on ξ only. This line of argument was originally called 'the dimensional hypothesis', but it has become clear that no *hypothesis* whatever is involved; the substratum is a kinematic entity, whose properties must arise simply in virtue of the construction or definition of the substratum. In particular, that property of the substratum which is represented by the equation of motion of a free test-particle must be derivable, if at all, without reference to *constants of nature*, which play no part in the construction of the substratum.

73. Problem of the determination of $G(\xi)$. The next question is: Is it possible to go further and actually ascertain the form of $G(\xi)$ in (23), (24), and (26)?

A little consideration shows that it is not so possible without considering systems of particles in motion more general than a substratum. In deriving (23), (24), and **(26)** we have used the circumstance that the acceleration of a free test-particle must be of dimensions one in length and minus two in time; and we have used the fact that from its definition all fundamental observers in the substratum are equivalent and in uniform relative motion. But we have not fully used the fact that there is at each position P in the substratum, at every epoch $t > 0$, a unique velocity $\mathbf{V_0} = \mathbf{P}/t$. We have used the fact that each fundamental observer O' in the substratum is equivalent to every other fundamental observer O', but equations of motion of the form (23), (24), and (26) will hold good for any system which preserves the equivalence of all the fundamental observers.

74. Hydrodynamical and statistical systems. Now the substratum in t-measure, as considered in the previous chapter, is essentially of hydrodynamical character. There is a unique velocity of flow at each observer, namely the recession velocity $\mathbf{V_0} = \mathbf{P}/t$, and the particle-density distribution satisfies the hydrodynamical equation of continuity. We can, however, consider more generalized systems, which bear the same relation to a substratum as a gas

bears to a liquid. We can consider a system such that at each point
P at epoch t, there is a velocity-distribution, and then arrange this
velocity-distribution so that it is described in the same way by every
fundamental observer. In the presence of such a distribution, the
equation of motion of a free test-particle will still retain the form (23),
(24), or (26), with presumably some function $G(\xi)$ depending on the
form of the velocity-distribution function. It is clear that until we
introduce the negative property of a substratum that it is *not* one
of these more general systems, we cannot expect to fix $G(\xi)$.

It is notoriously difficult to introduce a negative property, and so
to derive the form of $G(\xi)$ from the property that the substratum is
not a statistical system with a velocity-distribution at each point,
but a hydrodynamical system with a characteristic velocity of flow
at each point. The only method I have found capable of doing this
is to study statistical systems compatible with the equivalence of
all observers O, and then reduce the statistical systems by a limiting
process to the hydrodynamical substratum, or, alternatively, give
a physical interpretation of the acceleration deduced for statistical
systems.

75. Fixation of $G(\xi)$ for a substratum. The details of this analysis
are given in Part III below, Chapter IX. The result is to establish
that for a substratum, the function $G(\xi)$ is given by

$$G(\xi) \equiv -1. \tag{27}$$

Accordingly the equations of motion of a free test-particle in the
presence of the substratum (23), (24), and (26) take the form, in
4-vectors,

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(\mathbf{P} - \mathbf{V}\frac{Z}{Y}\right), \tag{28}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(ct - c\frac{Z}{Y}\right), \tag{28'}$$

which in turn are equivalent to the 3-vector equation

$$\frac{d\mathbf{V}}{dt} = -\frac{Y}{X}(\mathbf{P} - \mathbf{V}t). \tag{29}$$

As before, (28') is the time-component equation corresponding to
(28), and between (28) and (28') there is one identical relation, so
that (28') is deducible from (28).

76. An integral of the equations of motion. We shall now show that

$$\xi^{\frac{1}{2}} = \text{const.} \tag{30}$$

is an integral of the equation of motion of a free particle. We have, since

$$\xi^{\frac{1}{2}} = Z/X^{\frac{1}{2}}Y^{\frac{1}{2}},$$

by direct differentiation

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\xi^{\frac{1}{2}}}{dt} = \frac{\xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\left\{ \frac{1}{Z}\frac{dZ}{dt} - \frac{Z}{X} + \left(\frac{\mathbf{V}}{c^2}\cdot\frac{d\mathbf{V}}{dt}\right)\frac{1}{Y}\right\}.$$

Now

$$\frac{dZ}{dt} = 1 - \frac{\mathbf{V}^2}{c^2} - \frac{\mathbf{P}}{c^2}\cdot\frac{d\mathbf{V}}{dt} = Y - \frac{\mathbf{P}}{c^2}\cdot\frac{d\mathbf{V}}{dt}.$$

Hence

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\xi^{\frac{1}{2}}}{dt} = \frac{\xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\left\{ \frac{Y}{Z} - \frac{Z}{X} - \frac{1}{c^2}\left(\frac{\mathbf{P}}{Z} - \frac{\mathbf{V}}{Y}\right)\cdot\frac{d\mathbf{V}}{dt}\right\}.$$

Substitute in this for $d\mathbf{V}/dt$ from (29). We get

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\xi^{\frac{1}{2}}}{dt} = \xi^{\frac{1}{2}}\left\{ \frac{1-\xi}{\xi^{\frac{1}{2}}X^{\frac{1}{2}}} + \frac{Y^{\frac{1}{2}}}{Xc^2}\left(\frac{\mathbf{P}^2-(\mathbf{P}\cdot\mathbf{V})t}{Z} - \frac{(\mathbf{P}\cdot\mathbf{V})-\mathbf{V}^2 t}{Y}\right)\right\}$$

$$= \xi^{\frac{1}{2}}\left\{ \frac{1-\xi}{\xi^{\frac{1}{2}}X^{\frac{1}{2}}} + \frac{(t^2-X)-t(t-Z)}{X^{\frac{3}{2}}\xi^{\frac{1}{2}}} - \frac{(t-Z)-t(1-Y)}{XY^{\frac{1}{2}}}\right\}$$

$$= \xi^{\frac{1}{2}}\left(\frac{1}{X^{\frac{1}{2}}\xi^{\frac{1}{2}}} - \frac{\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}} - \frac{1}{\xi^{\frac{1}{2}}X^{\frac{1}{2}}} + \frac{tY^{\frac{1}{2}}}{X} + \frac{\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}} - \frac{tY^{\frac{1}{2}}}{X}\right)$$

$$= 0.$$

It follows that $\xi^{\frac{1}{2}}$ is constant along the trajectory of a free particle. We shall now define the *inertial mass* of a particle moving through the position P at epoch t with velocity $\mathbf{V}$, relative to a fundamental observer O at the origin, as M given by

$$M = m\xi^{\frac{1}{2}}, \tag{31}$$

where m is a constant characteristic of the particle. Written out, this is

$$M = m\frac{t-\mathbf{P}\cdot\mathbf{V}/c^2}{(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}(t^2-\mathbf{P}^2/c^2)^{\frac{1}{2}}}. \tag{31'}$$

77. Some properties of the mass of a particle. Since $\xi^{\frac{1}{2}} = \text{const.}$ is an integral of the motion of a free particle, the mass M is constant along the path of a free particle. The mass of the fundamental particle at the origin is obtained by putting $\mathbf{P} = 0$, $\mathbf{V} = 0$ in (31'), and taking $m = m_0$, say, where m_0 is the constant characteristic of fundamental particles. The result is

$$M = m_0. \tag{32}$$

Now consider the mass of the fundamental particle at P_0 at epoch t, moving accordingly with speed $V_0 = P_0/t$. Putting $V = V_0 = P_0/t$ in (31') we get again

$$M = m_0. \tag{33}$$

Thus the masses of all fundamental particles are equal.

Now consider the mass of a particle at the origin moving with speed V. Putting $P = 0$ in (31') we get

$$M = \frac{m}{(1-V^2/c^2)^{\frac{1}{2}}}. \tag{34}$$

This formally agrees with Einstein's definition of mass. But there is an essential difference between (34) and Einstein's formula. In the present work, since ξ^{i} is an invariant, the mass M is an invariant, taking the same value whatever fundamental observer O is chosen. Relation (34) gives the particular case when the fundamental observer O coincides, in position but not of course in velocity, with the particle concerned. In Einstein's formula, on the other hand, the frame of reference and accordingly the velocity V are arbitrary; the mass is effectively the fourth component of a 4-vector, and can take any numerical value whatever depending on choice of frame of reference. In the present dynamics the mass of a particle is perfectly definite, and independent of whatever fundamental frame of reference is chosen. For example, in Einstein's mechanics, distant fundamental particles, moving with speeds approaching c, would be regarded as possessing very large masses; in the present dynamics, all fundamental particles have the same mass. It will appear in the next chapter that with an appropriate definition of rate of performance of work, the energy E of a free particle in motion is given by $E = Mc^2$. This again, though formally identical with Einstein's mass-energy relation, must be sharply distinguished from it. For on Einstein's mechanics, distant fundamental particles have large stores of kinetic energy, owing to their motion of recession; on the present mechanics, they have no more energy than a fundamental particle anywhere.

Again, consider a particle at P at epoch t at rest relative to the observer O at the origin, so that $V = 0$. Then (31') gives

$$M = \frac{m}{(1-P^2/c^2t^2)^{\frac{1}{2}}}. \tag{35}$$

The excess of this over m can be regarded if we like as potential energy due to the gravitational field of the substratum. For with

our subsequent definition of work it can be shown to be $1/c^2$ times the work required to bring it from rest at the origin to rest at P at epoch t. But it is perhaps preferable to notice that since $\mathbf{P}/t$ is the velocity of the fundamental particle at P, (35) may be written in the form

$$M = \frac{m}{(1 - \mathbf{V}_0^2/c^2)^{\frac{1}{2}}}, \qquad (35')$$

which exhibits the mass of the particle at rest at P relative to the observer at O as arising from its velocity relative to the fundamental particle at P, which is moving with velocity $\mathbf{V}_0 = \mathbf{P}/t$. The fundamental observer O at the origin regards the particle in question as at rest, and its energy accordingly as solely potential, but the fundamental observer $\mathbf{P}$ in the vicinity of the particle in question regards its energy as kinetic, arising from its motion. Thus mass or energy may be regarded at will as purely potential or purely kinetic, depending on choice of observer.

It is readily shown that $\xi^4 > 1$ unless the circumstances of motion are those of a fundamental particle. For we shall have $\xi^4 > 1$ provided

$$Z^2 > XY,$$

or

$$(t - \mathbf{P}.\mathbf{V}/c^2)^2 > (t^2 - \mathbf{P}^2/c^2)(1 - \mathbf{V}^2/c^2),$$

or

$$t^2(\mathbf{V}^2/c^2) - 2t(\mathbf{P}.\mathbf{V}/c^2) + \mathbf{P}^2/c^3 - \{\mathbf{P}^2\mathbf{V}^2 - (\mathbf{P}.\mathbf{V})^2\}/c^4 > 0.$$

This quadratic in t will be always positive provided

$$(\mathbf{P}.\mathbf{V}/c^2)^2 < (\mathbf{V}^2/c^2)[\mathbf{P}^2/c^3 - \{\mathbf{P}^2\mathbf{V}^2 - (\mathbf{P}.\mathbf{V})^2\}/c^4],$$

or

$$(\mathbf{P}\wedge\mathbf{V})^2(1 - \mathbf{V}^2/c^2) > 0.$$

This condition is satisfied unless $\mathbf{V}$ is parallel to $\mathbf{P}$. In that case the quadratic in t reduces to

$$(\mathbf{V}t - \mathbf{P})^2 > 0,$$

which is satisfied save when $\mathbf{V} = \mathbf{P}/t$, i.e. when the motion is that of a fundamental particle. In that case $\xi^4 = 1$.

78. Variational principle for a free particle.

The equations of motion (28) and (28'), or (29), can be shown to be reproduced by either of the variational principles

$$\delta \int \frac{Y^{\frac{1}{2}}dt}{X^{\frac{1}{2}}} = 0, \quad \text{or} \quad \delta \int \frac{Y}{Z} dt = 0. \qquad (36), (36')$$

We first notice that from their form, each integrand is a 4-scalar, and therefore the resulting equations of motion will be Lorentz-invariant.

Consider now the variational principle (36). The corresponding Eulerian equations are

$$\frac{d}{dt}\left(\frac{\partial}{\partial V}\left(\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\right) - \frac{\partial}{\partial P}\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} = 0,$$

which gives

$$\frac{d}{dt}\left(-\frac{V}{X^{\frac{1}{2}}Y^{\frac{1}{2}}}\right) - \frac{PY^{\frac{1}{2}}}{X^{\frac{1}{2}}} = 0.$$

Or

$$-\frac{1}{X^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right) + \frac{V}{X^{\frac{3}{2}}}\frac{Z}{Y^{\frac{1}{2}}} - P\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} = 0,$$

or

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(P - V\frac{Z}{Y}\right), \tag{37}$$

which is (28). The variational principle (36′) follows from the fact that it may be written in the form

$$\delta \int \frac{1}{\xi^{\frac{1}{2}}}\frac{Y^{\frac{1}{2}}\,dt}{X^{\frac{1}{2}}} = 0, \tag{36″}$$

since $\xi^{\frac{1}{2}} = $ const. is an integral of (36). It can also be shown that (37) may be derived from (36″).

79. Interpretation of the equation of motion of a free particle.

The equation of motion (29) can be integrated completely[†] and the corresponding trajectory found in t-measure. It is of more significance, however, to proceed now to derive the corresponding equation of motion for the relatively stationary equivalence, and to interpret the latter physically.

Previously, when we have considered the regraduation of clocks, we have named the clock-readings t before regraduation and T afterwards. We shall henceforth use τ to denote the regraduated clock-reading in the relatively stationary equivalence, and to denote also a typical time-coordinate derived from measures with the regraduated clocks. Thus the regraduation which transforms the uniform relative motion equivalence into the relatively stationary equivalence is

$$\tau = t_0\log(t/t_0) + t_0. \tag{38}$$

Here t_0 is the epoch t at which $\tau = t$; and at the same epoch, since (38) gives

$$\frac{d\tau}{t_0} = \frac{dt}{t}, \tag{39}$$

we have

$$d\tau = dt.$$

Thus at $t = \tau = t_0$, the τ- and t-clocks agree in both epoch and rate.

† *Proc. Roy. Soc.* **154** A, 48, 1936.

If the event $(t, \mathbf{P})$ in t-measure is described as $(\tau, \mathbf{\Pi})$ in τ-measure, then we have from the light-signals $t_1 \to t_2 \to t_3$, $\tau_1 \to \tau_2 \to \tau_3$,

$$|\mathbf{P}| = \tfrac{1}{2}c(t_3 - t_1), \qquad t = \tfrac{1}{2}(t_3 + t_1),$$

$$|\mathbf{\Pi}| = \tfrac{1}{2}c(\tau_3 - \tau_1), \qquad \tau = \tfrac{1}{2}(\tau_3 + \tau_1),$$

whence approximately for distances $|\mathbf{P}|$ small compared with ct we have

$$|\mathbf{P}| = \frac{\Delta t}{\Delta \tau} |\mathbf{\Pi}| = \frac{t}{t_0} |\mathbf{\Pi}|,$$

so that

$$\mathbf{P} = \frac{t}{t_0} \mathbf{\Pi}. \tag{40}$$

Hence

$$\mathbf{V} = \frac{d\mathbf{P}}{dt} = \frac{\mathbf{\Pi}}{t_0} + \frac{t}{t_0}\frac{d\mathbf{\Pi}}{dt} = \frac{\mathbf{\Pi}}{t_0} + \frac{d\mathbf{\Pi}}{d\tau},$$

or

$$\mathbf{V} - \frac{\mathbf{P}}{t} = \frac{d\mathbf{\Pi}}{d\tau}. \tag{41}$$

Thus, as seen before, the velocity relative to the neighbouring fundamental particle, $\mathbf{V} - \mathbf{P}/t$, becomes the velocity $d\mathbf{\Pi}/d\tau$ in the relatively stationary substratum.

For speeds $|\mathbf{V}|$ small compared with c, and distances $|\mathbf{P}|$ small compared with ct, the equation of motion (29) reduces approximately to

$$\frac{d\mathbf{V}}{dt} = -\frac{\mathbf{P} - \mathbf{V}t}{t^2}, \tag{42}$$

or

$$\frac{d}{dt}\left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = 0. \tag{43}$$

Referred to the relatively stationary equivalence, this becomes

$$\frac{d}{d\tau}\left(\frac{d\mathbf{\Pi}}{d\tau}\right) = 0. \tag{43'}$$

Thus the approximate equation of motion in τ-measure is simply the statement

$$acceleration = 0, \tag{44}$$

where the acceleration is measured relative to the relatively stationary substratum. But this is the Newtonian equation of motion of a free particle in empty space. It follows at once that τ is the time-variable of Newtonian physics.

This is a most important result. It shows that in the Newtonian scale of time the fundamental particles, which correspond to the nuclei of the galaxies, are to be considered as at rest. It shows also, since τ is a public time, the same (for a given event) for all

fundamental observers, that Newtonian time gives an absolute simultaneity throughout the universe. It shows that in Newtonian measures there is an absolute standard of rest throughout the universe, and that the universe extends to infinity in all directions, and contains an infinite number of extra-galactic nebular nuclei.

80. Acceleration in t-measure as a gravitational effect. The transformation of clocks (38) has, further, transformed away the acceleration on the right-hand side of the approximate equation of motion (42). This acceleration, the value of $d\mathbf{V}/dt$ in t-measure, may be considered as the gravitational pull of the substratum on the free test-particle. The vector $\mathbf{P} - \mathbf{V}t$ is the distance of the particle P from the apparent centre of the substratum in the frame in which P is at rest. Calling this distance r, the approximate value of the acceleration in the substratum is

$$\frac{dV}{dt} = -\frac{r}{t^2},$$

where t as usual is the epoch of the event $(\mathbf{P}, t)$ measured from the natural zero of time, i.e. t is the age of the system to the observer at O. Let us identify this acceleration with the Newtonian attraction that would be calculated for the vicinity of the centre of the substratum in the frame in which P is at rest. Since the particle density of the substratum in t-measure at the observer O is, by (5), Chap. IV, B/c^3t^3, the Newtonian attraction at distance r is that due to a spherical mass of amount

$$\frac{\frac{4}{3}\pi m_0\, Br^3}{c^3t^3}.$$

If γ is the Newtonian 'constant' of attraction, then equating the results of the kinematically calculated acceleration to the classically calculated acceleration, we get

$$-\frac{\gamma}{r^2}\frac{\frac{4}{3}\pi m_0\, Br^3}{c^3t^3} = -\frac{r}{t^2},$$

This gives

$$\gamma = \frac{c^3t}{\frac{4}{3}\pi m_0\, B}.$$

Thus the Newtonian 'constant' should be actually variable in time, increasing proportionally to the epoch. The mass $\frac{4}{3}\pi m_0\, B$ has a simple physical interpretation. If we take a sphere of radius equal to the radius of the universe, ct, in t-measure, and fill it with matter, homo-

geneously, of the same density as the mean density of the universe near ourselves, we get a total mass of

$$\tfrac{4}{3}\pi(ct)^3 \times \frac{m_0 B}{c^3 t^3} = \tfrac{4}{3}\pi m_0 B = M_0,$$

say.   This gives

$$\gamma = \frac{c^3 t}{M_0}.$$

M_0 is the mass of the equivalent homogeneous universe. (The substratum, in t-measure, is not of course homogeneous, and its true mass is infinite.) The numerical value of M_0 comes out at approximately the same number of grams as the number of grams usually assigned to the actual universe in the 'general relativity' theory. Taking $t = 2 \times 10^9$ years, $= 2 \times 10^9 \times 3.15 \times 10^7$ seconds, and the present value of γ as 6.66×10^{-8} C.G.S. units, and the value of c as 3×10^{10} cm. sec.$^{-1}$, the value of M_0 is given by

$$M_0 = \frac{c^3 t}{\gamma} = \frac{27 \times 10^{30} \times 2 \times 10^9 \times 3.15 \times 10^7}{6.66 \times 10^{-8}} = 2.55 \times 10^{55} \text{ grams.}$$

We shall see later that when we come to transform the general equation of motion of a particle in a gravitational field from t-measure to τ-measure, then the 'constant' of gravitation appears in the form

$$\gamma_0 = \frac{c^3 t_0}{M_0},$$

and is thus a constant in the equations, although its numerical value depends on t_0.

81. It is a consequence of our analysis that the general homogeneous distribution of the nebulae in τ-measure exerts no net gravitational pull on a free test-particle. Thus when Newtonian time is employed, a particle at large amongst the galaxies (save for the local gravitational effects of any nearby galaxies) moves as if in Newtonian 'empty space', with zero acceleration and so constant velocity.

So far we have dealt only with the approximate consequences of the regraduation of clocks for nearby particles moving with speeds not approaching c. We must now transform the equation of motion of a free particle rigorously. The simplest way of doing this is to employ one of the variation principles already established, (36) or (36').

82. Transformation of equation of motion of a free particle to τ-measure. We have seen in Chapter IV that the complete

transformation of coordinates between t-measure and τ-measure is given by

$$t = t_0\, e^{(\tau - t_0)/t_0} \cosh \lambda/ct_0, \tag{45}$$

$$r = ct_0\, e^{(\tau - t_0)/t_0} \sinh \lambda/ct_0. \tag{46}$$

We have also seen that when O adopts the private Euclidean space de^2 given by

$$de^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{47}$$

with public space-time ds^2 given by

$$ds^2 = dt^2 - de^2/c^2, \tag{48}$$

then in τ-measure he must use a public hyperbolic space $d\epsilon^2$ and space-time $d\sigma^2$ given by

$$ds = e^{(\tau - t_0)/t_0} d\sigma, \tag{49}$$

$$d\sigma^2 = d\tau^2 - d\epsilon^2/c^2 = d\tau^2(1 - v^2/c^2), \tag{50}$$

$$d\epsilon^2 = d\lambda^2 + (ct_0)^2\sinh^2(\lambda/ct_0)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{51}$$

But (45) and (46) give

$$X^{\tfrac{1}{2}} = t_0\, e^{(\tau - t_0)/t_0}, \tag{52}$$

whilst

$$Y^{\tfrac{1}{2}}\, dt = ds. \tag{53}$$

Hence

$$\frac{Y^{\tfrac{1}{2}}\, dt}{X^{\tfrac{1}{2}}} = \frac{d\sigma}{t_0}, \tag{54}$$

and the variational principle (36), namely

$$\delta \int \frac{Y^{\tfrac{1}{2}}\, dt}{X^{\tfrac{1}{2}}} = 0, \tag{55}$$

gives

$$\delta \int d\sigma = 0, \tag{56}$$

or

$$\delta \int \left(1 - \frac{v^2}{c^2}\right)^{\tfrac{1}{2}} d\tau = 0, \tag{57}$$

where v^2 is the square of the velocity in the hyperbolic space (51), namely

$$v^2 = \left(\frac{d\epsilon}{d\tau}\right)^2. \tag{58}$$

Similarly we can use the variational principle (36'). We have, since

$$\frac{dX}{dt} = 2Z,$$

the equality

$$\frac{1}{Y^{\tfrac{1}{2}}}\frac{dX^{\tfrac{1}{2}}}{dt} = \xi^{\tfrac{1}{2}}.$$

But, by (52),
$$dX^{\frac{1}{2}} = e^{(\tau - t_0)/t_0} d\tau,$$
(59)

and, by (53) and (49),
$$Y^{\frac{1}{2}} dt = e^{(\tau - t_1)/t_0} d\sigma.$$
(60)

Hence, by (59) and (60),
$$\xi^{\frac{1}{2}} = \frac{d\tau}{d\sigma} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$
(61)

Hence the variational principle (36'), namely
$$\delta \int \frac{Y}{Z} dt = 0 \quad \text{or} \quad \delta \int \xi^{-\frac{1}{2}} \frac{Y^{\frac{1}{2}} dt}{X^{\frac{1}{2}}} = 0,$$

gives
$$\delta \int \frac{d\sigma}{d\tau} d\sigma = 0,$$

or
$$\delta \int \left(1 - \frac{v^2}{c^2}\right) d\tau = 0.$$
(62)

Result (56) shows that the path of a free particle in the hyperbolic space-time (50) is a geodesic. Result (62) gives for the corresponding Eulerian equations
$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\lambda}}\right) - \frac{\partial T}{\partial \lambda} = 0,$$
(63)

$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0,$$
(63')

$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = 0,$$
(63")

where
$$T = \tfrac{1}{2}mv^2.$$

We can always choose O so that initially $\theta = 0$, $\phi = 0$, $\dot{\theta} = 0$, $\dot{\phi} = 0$. Then (63') and (63") show that θ and ϕ remain permanently zero, and (63) gives then
$$\frac{d^2\lambda}{d\tau^2} = 0.$$
(64)

Thus reckoned by a fundamental observer in the track of the free particle, the free particle moves with zero acceleration in the hyperbolic space $d\epsilon^2$. This establishes the Galileo-Newton principle of inertia for τ-time and the associated hyperbolic space, and again identifies the time τ as the uniform time of Newtonian physics.

83. Reasons for deriving equations of motion first in t-measure.

It may be asked why we did not attempt to derive the equation of motion of a free particle directly for τ-measure instead of deriving it first in t-measure and then transforming. The answer is that it

seems impossible to discover a compelling argument why the acceleration in τ-measure should be *a priori* zero. There is no natural zero of time in τ-measure, and so it is true we cannot expect the time-variable τ to enter into the expression for the acceleration $d^2\lambda/d\tau^2$. But there enters into the description of the substratum in τ-measure a parameter t_0, and the acceleration of a free particle might *a priori* be proportional to λ/t_0^2. No such parameter enters into the description of the substratum in t-measure; instead, we have the coordinate t, which plays, as we have seen, an important part in the equation of motion in t-measure. Again, the rules of transformation from one fundamental observer to another are less powerful when the fundamental particles are relatively stationary than when they are in uniform relative motion away from one another. Lastly, any direct argument purporting to establish $d^2\lambda/d\tau^2 = 0$ would be rightly suspect, on the ground that the result of the argument was already known beforehand. The equation of motion in t-measure, on the other hand, was obtained long before its interpretation in τ-measure had been obtained.

VI

CONSTRUCTION OF A DYNAMICS

84. Object of the chapter. Once the equation of motion of a free particle has been obtained it is possible to proceed to the construction of a dynamics. That is to say, if a particle's motion does not coincide with the motion of a free particle, the difference between the two motions allows us to introduce a measure of the *force* acting on the particle, which is then regarded as constrained. And once we have succeeded in defining *force*, we can proceed to define a field of potential, and the rate of performance of work on the particle. We shall carry out this programme, first in t-measure, and later transform our results into τ-measure.

85. Definition of force. We have seen that the equations of motion of a free particle, in the form

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(P - V\frac{Z}{Y}\right), \tag{1}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(ct - c\frac{Z}{Y}\right), \tag{1'}$$

or in the equivalent form

$$\frac{dV}{dt} = -\frac{Y}{X}(P - Vt), \tag{2}$$

possess the integral $\qquad \xi^{\frac{1}{2}} = \text{const.}, \tag{3}$

and we have identified the number $m\xi^{\frac{1}{2}}$ as the *mass* of the particle moving through position P at epoch t with velocity V, as reckoned by a fundamental observer O at the origin. It therefore suggests itself that, as we want our t-dynamics to resemble classical dynamics as far as possible, we should define the force 4-vector (F, F_t) acting on a particle whose acceleration does not coincide with the acceleration of a free test-particle at the same position at the same epoch moving with the same velocity, by the equations

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(P - V\frac{Z}{Y}\right) + F, \tag{4}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{c}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(ct - c\frac{Z}{Y}\right) + F_t. \tag{4'}$$

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It is not difficult to verify that when we put $\mathbf{F} = 0$, $F_t = 0$, the resulting equations possess the integral $\xi^{\natural} = $ const., and so reduce to the equations of motion of a free particle, (1), (1'). For, if we put $\mathbf{F} = 0$ in (4), it reduces to

$$\frac{1}{\xi^{\natural}}\frac{d\xi^{\natural}}{dt}\frac{\mathbf{V}}{Y^{\natural}} + \frac{1}{Y^{\natural}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\natural}}\right) = -\frac{1}{X}\left(\mathbf{P} - \mathbf{V}\frac{Z}{Y}\right).$$

This implies the relation, obtained also by putting $F_t = 0$ in (4'),

$$\frac{1}{\xi^{\natural}}\frac{d\xi^{\natural}}{dt}\frac{c}{Y} + \frac{1}{Y^{\natural}}\frac{d}{dt}\left(\frac{c}{Y^{\natural}}\right) = -\frac{1}{X}\left(ct - c\frac{Z}{Y}\right).$$

Multiplying the first of these scalarly by $\mathbf{V}/c^2$ we get

$$\frac{1}{\xi^{\natural}}\frac{d\xi^{\natural}}{dt}\frac{1-Y}{Y} + \frac{1}{Y^2}\left(\frac{\mathbf{V}}{c^2}\cdot\frac{d\mathbf{V}}{dt}\right) = -\frac{1}{X}\left(t - \frac{Z}{Y}\right).$$

But the second is simply

$$\frac{1}{\xi^{\natural}}\frac{d\xi^{\natural}}{dt}\frac{1}{Y} + \frac{1}{Y^2}\left(\frac{\mathbf{V}}{c^2}\cdot\frac{d\mathbf{V}}{dt}\right) = -\frac{1}{X}\left(t - \frac{Z}{Y}\right).$$

Subtracting the last two equations we get

$$\frac{1}{\xi^{\natural}}\frac{d\xi^{\natural}}{dt} = 0.$$

Hence $\mathbf{F} = 0$, $F_t = 0$ imply the equations of motion of a free particle.

86. Relation between F and F_t. Moreover, since $\mathbf{F} = 0$ implies $F_t = 0$, there must be an identical relation between $\mathbf{F}$ and F_t. To find this, multiply (4) scalarly by $\mathbf{V}/c^2$. We got

$$\frac{1-Y}{Y}\frac{d}{dt}(m\xi^{\natural}) + \frac{m\xi^{\natural}}{Y^2}\left(\frac{\mathbf{V}}{c^2}\cdot\frac{d\mathbf{V}}{dt}\right) = -\frac{m\xi^{\natural}}{X}\left(t - \frac{Z}{Y}\right) + \frac{\mathbf{F}\cdot\mathbf{V}}{c^2}.$$

But (4') gives

$$\frac{1}{Y}\frac{d(m\xi^{\natural})}{dt} + \frac{m\xi^{\natural}}{Y^2}\left(\frac{\mathbf{V}}{c^2}\cdot\frac{d\mathbf{V}}{dt}\right) = -\frac{m\xi^{\natural}}{X}\left(t - \frac{Z}{Y}\right) + \frac{F_t}{c}.$$

Comparing the last two equations, we have

$$F_t\frac{c}{Y^{\natural}} - \mathbf{F}\cdot\frac{\mathbf{V}}{Y^{\natural}} = \frac{1}{Y^{\natural}}\frac{d}{dt}(mc^2\xi^{\natural}). \tag{5}$$

This relation must not be confused with the relation giving the rate of performance of work by the 4-vector force $(\mathbf{F}, F_t)$, about to be obtained. It should be noted, in fact, that in deriving (5) we have not used the definition of $\xi^{\natural}$ in terms of $\mathbf{P}$, t, $\mathbf{V}$.

87. Rate of performance of work by $(\mathbf{F}, F_l)$. We want to define a 4-scalar which will represent the rate of performance of work by the 4-vector force $(\mathbf{F}, F_l)$ in pushing the constrained particle relative to its immediate cosmic environment. To do this we need a 4-vector to represent the velocity of the particle relative to its immediate cosmic environment in the substratum. Consider the velocity-derivative of $\Omega = mc^2 \xi^{\frac{1}{2}}$. We have

$$Y^{\frac{1}{2}} \frac{\partial \Omega}{\partial \mathbf{V}} = m Y^{\frac{1}{2}} \xi^{\frac{1}{2}} \left(-\frac{\mathbf{P}}{Z} + \frac{\mathbf{V}}{Y} \right) = m \xi^{\frac{1}{2}} \left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z} \right). \tag{6}$$

We shall show that Ω represents the kinetic energy of the particle, and therefore its 4-vector $\mathbf{V}$-derivative may be expected to represent in some way its momentum. Scrutiny of the right-hand side shows that it vanishes for a fundamental particle, namely for $\mathbf{V} = \mathbf{P}/t$. Since $m \xi^{\frac{1}{2}}$ represents mass, we can take

$$\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z}, \quad \frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right) \tag{7}$$

to be the 4-vector representing the velocity of the particle relative to its immediate environment in the substratum. We shall accordingly adopt as a definition of the rate of performance of work by the 4-vector $(\mathbf{F}, F_l)$ representing force, the scalar product

$$\mathbf{F} \cdot \left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z} \right) - F_l \left(\frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right), \tag{8}$$

and we shall call this

$$\frac{1}{Y^{\frac{1}{2}}} \frac{dW}{dt}. \tag{9}$$

Substituting in expression (8) for $\mathbf{F}$ and F_l from (4) and (4'), we have

$$\frac{1}{Y^{\frac{1}{2}}} \frac{dW}{dt} = \frac{m \xi^{\frac{1}{2}}}{X} \left\{ \left(\mathbf{P} - \mathbf{V} \frac{Z}{Y} \right) \cdot \left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z} \right) - \left(ct - c \frac{Z}{Y} \right) \left(\frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right) \right\} +$$
$$+ \left\{ \left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z} \right) \cdot \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt} \left(m \xi^{\frac{1}{2}} \frac{\mathbf{V}}{Y^{\frac{1}{2}}} \right) - \left(\frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right) \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt} \left(m \xi^{\frac{1}{2}} \frac{c}{Y^{\frac{1}{2}}} \right) \right\}.$$

On using the definitions of X, Y, and Z, we find that the first term on the right-hand side reduces to

$$m \xi^{\frac{1}{2}} \frac{c^2}{X} \left(\frac{X Y^{\frac{1}{2}}}{Z} - \frac{Z}{Y^{\frac{1}{2}}} \right),$$

whilst the second term on the right-hand side reduces to

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt} \left(\frac{m \xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}} \right) \left(-\frac{Y}{Y} + \frac{Y^{\frac{1}{2}}}{Z} \frac{Z}{Y^{\frac{1}{2}}} \right) + \frac{m \xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}} \left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P} \frac{Y^{\frac{1}{2}}}{Z} \right) \cdot \frac{1}{Y^{\frac{1}{2}}} \frac{d\mathbf{V}}{dt}.$$

Hence

$$\frac{1}{Y^{\frac{1}{2}}}\frac{dW}{dt} = mc^2\xi^{\frac{1}{2}}\left(\frac{Y^{\frac{1}{2}}}{Z} - \frac{Z}{XY^{\frac{1}{2}}}\right) + \frac{m\xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\left(\frac{V}{Y^{\frac{1}{2}}} - P\frac{Y^{\frac{1}{2}}}{Z}\right)\cdot\frac{1}{Y^{\frac{1}{2}}}\frac{dV}{dt}.$$

But by direct differentiation we have

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) = \frac{m\xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\left(\frac{Y}{Z} - \frac{Z}{X} - \frac{P}{Z}\cdot\frac{dV}{dt} + \frac{V}{Y}\cdot\frac{dV}{dt}\right).$$

Hence
$$\frac{1}{Y^{\frac{1}{2}}}\frac{dW}{dt} = \frac{1}{Y^{\frac{1}{2}}}\frac{d(mc^2\xi^{\frac{1}{2}})}{dt},$$

or
$$\mathbf{F}\cdot\left(\frac{V}{Y^{\frac{1}{2}}} - P\frac{Y^{\frac{1}{2}}}{Z}\right) - F_t\left(\frac{c}{Y^{\frac{1}{2}}} - ct\frac{Y^{\frac{1}{2}}}{Z}\right) = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}). \qquad (10)$$

Since the left-hand side of this represents the rate of performance of work done in pushing the particle relative to its surroundings in the substratum, the right-hand side represents the rate of gain of energy of the particle. It follows that, as anticipated,

$$\Omega = mc^2\xi^{\frac{1}{2}} \qquad (11)$$

represents the *energy* of the particle.

88. The substratum as the seat of a conservative field of force.
Since $\xi^{\frac{1}{2}} = $ const. is an integral of the equation of motion of a free particle, it follows that the energy of a *free* particle remains constant during its motion. Further, if along two trajectories we have $\xi^{\frac{1}{2}} = \xi^{\frac{1}{2}}_1$ and $\xi^{\frac{1}{2}} = \xi^{\frac{1}{2}}_2$, then the difference $mc^2(\xi^{\frac{1}{2}}_2 - \xi^{\frac{1}{2}}_1)$ represents the work required to transport the particle from the first trajectory to the second. This is therefore independent of the path followed. The substratum therefore possesses a property analogous to the conservative field of force of classical mechanics, namely that if a particle in free motion is constrained so as to move from its trajectory along any 'circuit' back to the continuation of the trajectory, the work done by the constraining forces is zero.

89. Relation between mass and energy. In terms of P, t, V the energy of a particle is the scalar

$$\Omega = mc^2\frac{t - P\cdot V/c^2}{(1 - V^2/c^2)^{\frac{1}{2}}(t^2 - P^2/c^2)^{\frac{1}{2}}} = Mc^2. \qquad (12)$$

Thus the energy of a particle due to its position and velocity is c^2 times its mass—a relation first found by Einstein. There is, however, this difference, that in the present t-dynamics energy and mass are both 4-scalars, whilst in Einstein's mechanics they are represented

by the fourth or time-component of a 4-vector of which the first three components represent the momentum. Accordingly, in Einstein's dynamics the energy is not definite until an inertial frame has been selected, and the value of the energy depends on the inertial frame selected; whereas in the present t-dynamics the energy is the same for all fundamental observers, who represent equivalent frames of reference.

90. Properties of the expression for energy. Just as for mass, we can take different particular cases of formula (12). For a particle at rest at O, $V = 0$ and $P = 0$, and Ω reduces to mc^2. Ω also reduces to mc^2 for any fundamental particle, $V = P/t$. This is a highly satisfactory feature of the t-dynamics, for it means that the motions of the fundamental particles do not represent stores of kinetic energy. In its application to the universe of receding nebulae, this means that the huge outward velocities of the distant nebulae do not represent stores of kinetic energy.

91. When $P = 0$, Ω reduces to

$$\frac{mc^2}{(1-V^2/c^2)^{\frac{1}{2}}},\tag{13}$$

which coincides with Einstein's expression for kinetic energy. But it must be remembered that the V in this expression is the velocity relative to the observer at the origin and is not arbitrary. It is only the excess velocity of a particle relative to its immediate surroundings in the substratum which represents excess kinetic energy. Formula (12) may be written approximately as

$$\Omega = mc^2\left(1+\frac{1}{2}\frac{V^2}{c^2}+\frac{1}{2}\frac{P^2}{c^2t^2}-\frac{P.V}{c^2t}+\ldots\right)$$

$$= mc^2+\tfrac{1}{2}m\left(V-\frac{P}{t}\right)^2+\ldots,\tag{14}$$

which shows explicitly that kinetic energy is associated with the excess of the velocity V over the velocity P/t of the immediate surroundings in the substratum. In this connexion it is interesting to notice that if we put for the relative momentum

$$p_r = Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial V} = m\xi^{\frac{1}{2}}\left(\frac{V}{Y^{\frac{1}{2}}}-P\frac{Y^{\frac{1}{2}}}{Z}\right),$$

$$(p_r)_t = -Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial c} = m\xi^{\frac{1}{2}}\left(\frac{c}{Y^{\frac{1}{2}}}-ct\frac{Y^{\frac{1}{2}}}{Z}\right),$$

then
$$\mathbf{p}_r^2 - (p_r)_t^2 = m^2 c^2 \xi \left(1 - \frac{1}{\xi} \right) = \frac{\Omega^2}{c^2} - m^2 c^2,$$

or
$$\Omega^2 = m^2 c^4 + c^2 \{ \mathbf{p}_r^2 - (p_r)_t^2 \}. \tag{15}$$

For a fundamental particle both $\mathbf{p}_r$ and $(p_r)_t$ reduce to zero.

For $\mathbf{V} = 0$, $\mathbf{P} \neq 0$, we have

$$\Omega = \frac{mc^2}{\{1 - \mathbf{P}^2/(c^2 t)^2\}^{\frac{1}{2}}} = \frac{mc^2}{(1 - V_0^2/c^2)^{\frac{1}{2}}}, \tag{16}$$

where V_0 is the velocity of the fundamental particle with which the moving particle instantaneously coincides. This again shows that in (13) the origin of velocity is the fundamental particle in the vicinity of the moving particle.

92. Introduction of a potential function to represent an external force. We have seen that (5) represents the identical relation between the equations of motion (4) and (4') and that (10) is the expression for the rate of performance of 'external work' by the 'force'. These show at once that when $\mathbf{F} = 0$, then also $F_t = 0$ and ξ^t is constant, confirming that $\xi^t = \text{const.}$ is an integral of the motion of a free particle.

It suggests itself next that we try to define a conservative field of force χ, superposed on the substratum, by means of relations

$$\mathbf{F} = -\frac{\partial \chi}{\partial \mathbf{P}}, \qquad F_t = \frac{\partial \chi}{c \partial t},$$

in the hope that the two relations (5) and (10) will yield a relation

$$\frac{d}{dt}[\chi + \Omega] = 0. \tag{17}$$

But χ must be a homogeneous function of P and ct of such dimensions that χ/mc^2 is of dimensions zero. Hence, by Euler's theorem on homogeneous functions,

$$ct \frac{\partial \chi}{c \partial t} + \mathbf{P} \cdot \frac{\partial \chi}{\partial \mathbf{P}} = 0.$$

The two relations (5) and (10) then reduce to

$$\frac{d\chi}{dt} = \frac{d(mc^2 \xi^t)}{dt}, \qquad -\frac{d\chi}{dt} = \frac{d(mc^2 \xi^t)}{dt},$$

where
$$\frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + \mathbf{V} \cdot \frac{\partial \chi}{\partial \mathbf{P}},$$

and these are self-contradictory.

We have therefore to proceed on other lines. Consider the result of adding together the energy-integral (10) and the pseudo-energy integral (5). The result is

$$(F_t ct - \mathbf{F} . \mathbf{P}) \frac{Y^{\frac{1}{2}}}{Z} = \frac{2}{Y^{\frac{1}{2}}} \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}), \tag{18}$$

which may be written

$$F_t c - \mathbf{F} . \frac{\mathbf{P}}{t} = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}})\left(\frac{2Z}{tY}\right). \tag{19}$$

Compare this with the identical relation (5) between $\mathbf{F}$ and F_t previously obtained, namely

$$F_t c - \mathbf{F} . \mathbf{V} = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}).$$

Subtracting these two relations we have

$$\mathbf{F} . \left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}})\left(\frac{2Z}{tY} - 1\right).$$

But

$$\mathbf{V} . \left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = c^2\left(1 - Y - \frac{t-Z}{t}\right) = c^2 Y\left(\frac{Z}{tY} - 1\right).$$

Multiply the last equality by

$$\frac{2}{Y} \frac{d}{dt}(m\xi^{\frac{1}{2}}),$$

and subtract it from the preceding one. We get

$$\left\{\mathbf{F} - 2\frac{\mathbf{V}}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m\xi^{\frac{1}{2}})\right\} . \left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}). \tag{20}$$

This suggests that we put

$$\mathbf{F} = -\frac{\partial\chi}{\partial\mathbf{P}} + 2\frac{\mathbf{V}}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{21}$$

so that

$$F_t = +\frac{\partial\chi}{c\partial t} + 2\frac{c}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m\xi^{\frac{1}{2}}). \tag{21'}$$

We have then

$$\left(-\frac{\partial\chi}{\partial\mathbf{P}}\right) . \left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}). \tag{22}$$

We now examine the forms the energy-integral (10) and the pseudo-energy integral (5) take when $\mathbf{F}$, F_t are defined by (21), (21').

Inserting (21) and (21') in (5), we get

$$\left(\frac{Y}{Y^{\frac{1}{2}}}\right)\frac{2}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) + \frac{1}{Y^{\frac{1}{2}}}\left(\frac{\partial\chi}{\partial t} + \mathbf{V} . \frac{\partial\chi}{\partial\mathbf{P}}\right) = \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}),$$

or

$$\frac{d}{dt}(\chi + mc^2\xi^{\frac{1}{2}}) = 0. \tag{23}$$

Again, inserting (21) and (21′) in (10), we get

$$-\frac{1}{Y^{\frac{1}{2}}}\frac{d\chi}{dt}+\frac{Y^{\frac{1}{2}}}{Z}\left(\mathbf{P}\cdot\frac{\partial\chi}{\partial\mathbf{P}}+t\frac{\partial\chi}{\partial t}\right)+\frac{2}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}})\left(-\frac{Y}{Y}+\frac{Z}{Y^{\frac{1}{2}}}\frac{Y^{\frac{1}{2}}}{Z}\right)$$
$$=\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}).$$

The third term on the left-hand side vanishes identically. By (23), the first term on the left-hand side is equal to the right-hand side. We are left with

$$\mathbf{P}\cdot\frac{\partial\chi}{\partial\mathbf{P}}+t\frac{\partial\chi}{\partial t}=0, \tag{24}$$

which is satisfied identically if χ/mc^2 is a homogeneous function of P and ct of dimensions zero. Relations (5) and (10) are accordingly now consistent with the definitions (21) and (21′) of χ.

When $\mathbf{F}$, F_t are expressible in the forms (21), (21′) we say that the external force is derivable from a potential χ, a 4-scalar satisfying (24) identically. Relation (23) then shows that χ may be regarded as the potential energy of the particle with kinetic energy Ω or $mc^2\xi^{\frac{1}{2}}$. Also relation (22) has the interesting physical interpretation that the rate of increase of kinetic energy of the particle is equal to the 3-scalar work done by the gradient of the potential in pushing the particle with velocity $\mathbf{V}-\mathbf{P}/t$ relative to its immediate environment in the substratum. Moreover, (21) and (21′) show at once that when the external force $(\mathbf{F}, F_t)$ vanishes, so that $\Omega = mc^2\xi^{\frac{1}{2}} = $ const. is an integral, the gradients of the potential are zero, and there is no external field, as is required for consistency.

Evidently the second term on the right-hand sides of (21) and (21′) represents the effect of rate of change of mass M.

It must be emphasized that the actual details of the definition of a potential function are immaterial; when we derive a specific potential by kinematic methods, the actual form of the potential function will depend on the definition adopted, but when the resulting force-vectors are inserted in the equation of motion, the motion due to these force-vectors will be independent of the definition of potential adopted. Our main application of the theory of potential functions will occur when we derive the potential function for a pair of gravitating particles.

93. Occurrence of the factor 2 in the formula connecting force and potential gradient. The curious occurrence of the factor 2 in front of the 'rate of change of mass' term in (21) and (21′) should

be noted. Unnatural though it appears, it will be found in later developments to be essential. With the expressions (21) and (21') for F, F_l in terms of a 4-scalar potential χ, the equations of motion (4), (4') take the form, on differentiating out the left-hand sides,

$$\frac{m\xi^{\downarrow}}{Y^{\downarrow}}\frac{d}{dt}\left(\frac{V}{Y^{\downarrow}}\right) = -\frac{m\xi^{\downarrow}}{X}\left(P-V\frac{Z}{Y}\right)-\frac{\partial\chi}{\partial P}+\frac{V}{Y^{\downarrow}}\frac{1}{Y^{\downarrow}}\frac{d(m\xi^{\downarrow})}{dt}, \tag{25}$$

$$\frac{m\xi^{\downarrow}}{Y^{\downarrow}}\frac{d}{dt}\left(\frac{c}{Y^{\downarrow}}\right) = -\frac{m\xi^{\downarrow}}{X}\left(ct-c\frac{Z}{Y}\right)+\frac{1}{c}\frac{\partial\chi}{\partial t}+\frac{c}{Y^{\downarrow}}\frac{1}{Y^{\downarrow}}\frac{d}{dt}(m\xi^{\downarrow}). \tag{25'}$$

These are in 4-vector form. To obtain the associated 3-vector form, subtract V/c times the second (the scalar equation) from the first. We get

$$\frac{m\xi^{\downarrow}}{Y}\frac{dV}{dt} = -\frac{m\xi^{\downarrow}}{X}(P-Vt)-\left(\frac{\partial\chi}{\partial P}+\frac{V}{c^2}\frac{\partial\chi}{\partial t}\right). \tag{26}$$

94. Interpretation of the energy-integrals. The physical meaning of the energy-integral (10) and the pseudo-energy integral (5) is perhaps best seen by choosing $P = 0$, i.e. by taking the observer at the particle in question. Then they reduce to

$$F_l c-F.V = \frac{d\Omega}{dt}, \tag{27}$$

$$F.V-F_l c\left(\frac{V^2}{c^2}\right) = \frac{d\Omega}{dt}. \tag{28}$$

Solving for $F.V$ and $F_l c$, we get

$$\frac{F.V}{1+V^2/c^2} = \frac{F_l c}{2} = \frac{d\Omega/dt}{1-V^2/c^2}. \tag{29}$$

For small velocities, $|V| \ll c$, these give

$$F.V \sim d\Omega/dt, \qquad F_l c \sim 2d\Omega/dt. \tag{30}$$

Thus F_l is of the order of $2F.V/c$.

95. Alternative derivation. An alternative method of deriving the fundamental 'rate of performance of work' relation (10) may be of interest.

We have already defined the 'relative momentum' vector p_r by the relations

$$p_r = Y^{\downarrow}\frac{\partial\Omega}{\partial V} = M\left(\frac{V}{Y^{\downarrow}}-P\frac{Y^{\downarrow}}{Z}\right), \tag{31}$$

$$(p_r)_l = -Y^{\downarrow}\frac{\partial\Omega}{\partial c} = M\left(\frac{c}{Y^{\downarrow}}-ct\frac{Y^{\downarrow}}{Z}\right), \tag{32}$$

and shown that these satisfy relation (15), namely

$$c^2\{p_r^2-(p_r)_l^2\} = \Omega^2-m^2c^4. \tag{33}$$

This suggests another definition of external force, namely the 'apparent external force' $\{F_a, (F_a)_t\}$, by means of

$$F_a = \frac{1}{Y^{\frac{1}{2}}} \frac{dp_r}{dt}, \qquad (F_a)_t = \frac{1}{Y^{\frac{1}{2}}} \frac{d(p_r)_t}{dt}. \tag{34}$$

We shall now show that we have identically, if

$$p = m\xi^{\frac{1}{2}} \frac{V}{Y^{\frac{1}{2}}}, \qquad p_t = m\xi^{\frac{1}{2}} \frac{c}{Y^{\frac{1}{2}}}, \tag{35}$$

the relations

$$\frac{1}{Y^{\frac{1}{2}}} \frac{dp}{dt} \equiv -\frac{\partial \Omega}{\partial P} + F_a, \tag{36}$$

$$\frac{1}{Y^{\frac{1}{2}}} \frac{dp_t}{dt} \equiv +\frac{\partial \Omega}{c\partial t} + (F_a)_t. \tag{36'}$$

For these require

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{V}{Y^{\frac{1}{2}}}\right) \equiv -m\xi^{\frac{1}{2}}\left(\frac{P}{X} - \frac{V}{Z}\right) + \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left\{m\xi^{\frac{1}{2}}\left(\frac{V}{Y^{\frac{1}{2}}} - P\frac{Y^{\frac{1}{2}}}{Z}\right)\right\}, \tag{37}$$

with an analogous relation for the time-component. This identity will be satisfied if we can show that

$$m\xi^{\frac{1}{2}}\left(\frac{P}{X} - \frac{V}{Z}\right) \equiv -\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\xi^{\frac{1}{2}} P\frac{Y^{\frac{1}{2}}}{Z}\right).$$

But the right-hand side here is just

$$-\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\frac{P}{X^{\frac{1}{2}}}\right),$$

which is

$$\frac{m}{Y^{\frac{1}{2}}}\left(-\frac{V}{X^{\frac{1}{2}}} + P\frac{Z}{X^{\frac{1}{2}}}\right),$$

or

$$m\xi^{\frac{1}{2}}\left(\frac{P}{X} - \frac{V}{Z}\right).$$

We shall require identity (37) in the next chapter.

The difference between the external force (F, F_t) and the apparent external force $\{F_a, (F_a)_t\}$ performs zero work in any motion. For, subtracting the relations

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(P - V\frac{Z}{Y}\right) + F, \tag{38}$$

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(P - V\frac{X}{Z}\right) + F_a, \tag{39}$$

we get $$\mathbf{F}-\mathbf{F}_a = \frac{m\xi^{\frac{1}{2}}}{X}\left(\frac{X}{Z}-\frac{Z}{Y}\right)\mathbf{V} = -m\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{\xi-1}{X^{\frac{1}{2}}},\tag{40}$$

and similarly $$F_l-(F_a)_l = -m\frac{c}{Y^{\frac{1}{2}}}\frac{\xi-1}{X^{\frac{1}{2}}}.$$

The rate of performance of work by this difference is

$$-\frac{m(\xi-1)}{X^{\frac{1}{2}}}\left\{\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\cdot\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}-\mathbf{P}\frac{Y^{\frac{1}{2}}}{Z}\right)-\frac{c}{Y^{\frac{1}{2}}}\left(\frac{c}{Y^{\frac{1}{2}}}-ct\frac{Y^{\frac{1}{2}}}{Z}\right)\right\}$$

$$= -\frac{m(\xi-1)c^2}{X^{\frac{1}{2}}}\left(-\frac{Y}{Y}+\frac{Z}{Z}\right),$$

which vanishes identically. The rate of performance of work by the apparent external force is now given by

$$\frac{1}{Y^{\frac{1}{2}}}\frac{dW_a}{dt} = \frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{p}_r}{dt}\cdot\frac{\mathbf{p}_r}{M} - \frac{1}{Y^{\frac{1}{2}}}\frac{d(p_r)_l}{dt}\frac{(p_r)_l}{M}\tag{41}$$

$$= \frac{\frac{1}{2}}{MY^{\frac{1}{2}}}\frac{d}{dt}\{\mathbf{p}_r^2-(p_r)_l^2\},$$

or, using (15), $$\frac{1}{Y^{\frac{1}{2}}}\frac{dW_a}{dt} = \frac{\frac{1}{2}}{MY^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\Omega^2}{c^2}-m^2c^2\right).$$

But $Mc^2 = \Omega$. Hence

$$\frac{1}{Y^{\frac{1}{2}}}\frac{dW_a}{dt} = \frac{1}{Y^{\frac{1}{2}}}\frac{d\Omega}{dt},\tag{42}$$

and thus the rate of performance of work by the apparent external force is equal to the rate of increase of the kinetic energy Ω. It follows that the rate of performance of work by the external force, as defined by (4), (4'), is equal to the rate of increase of the kinetic energy Ω.

96. **Approximate formulae in τ-measure.** The next question which suggests itself is, How does the dynamics of a particle now constructed for t-measure go over into τ-measure? We shall transform it rigorously to τ-measure in the next chapter. For the immediately following sections we shall gain physical insight by making an approximate transformation from t-measure to τ-measure.

When $|\mathbf{P}| \ll ct$ and $|V| \ll c$, the equations of motion (4), (4') which define the external force $(\mathbf{F}, F_l)$ become

$$m\frac{d\mathbf{V}}{dt} = -m\frac{\mathbf{P}-\mathbf{V}t}{t^2}+\mathbf{F},\tag{43}$$

$$0 = F_l'.\tag{44}$$

The approximate transformation of coordinates, from t-measure to τ-measure, is given by

$$P = \Pi\frac{t}{t_0}, \qquad \frac{dt}{t} = \frac{d\tau}{t_0},$$

$$V = \frac{\Pi}{t_0} + \frac{t}{t_0}\frac{d\Pi}{dt} = \frac{P}{t} + \frac{d\Pi}{d\tau},$$

$$\frac{dV}{dt} = \frac{V}{t} - \frac{P}{t^2} + \frac{t_0}{t}\frac{d}{d\tau}\frac{d\Pi}{d\tau}.$$

Hence (43) becomes
$$m\frac{d^2\Pi}{d\tau^2} = \frac{t}{t_0}F.$$

Hence, if we define the external force in τ-measure, Φ, by

$$\Phi = \frac{t}{t_0}F, \qquad (45)$$

the equation of motion takes the form

$$m\frac{d^2\Pi}{d\tau^2} = \Phi. \qquad (46)$$

This is of the form of the Newtonian equation of motion if Φ is taken to be the actual external force when τ is the independent variable measuring time. It is readily seen that Φ is the actual transform of F to τ-measure. For, using the notation of dimensional equations, we have

$$\Phi = \frac{t}{t_0}F = \frac{t}{t_0}(m)\frac{(\Delta l)}{(\Delta t)^2} = (m)\frac{t}{t_0}\frac{\{(t/t_0)\Delta\lambda\}}{\{(t/t_0)\Delta\tau\}^2} = (m)\frac{(\Delta\lambda)}{(\Delta\tau)^2},$$

which shows that Φ is the actual measure of F on the τ-scale.

97. Kinetic energy—approximate form in τ-measure.

We now see that the kinetic energy Ω becomes in τ-measure

$$\Omega = mc^2 + \frac{1}{2}(V - P/t)^2 m = mc^2 + \frac{1}{2}(d\Pi/d\tau)^2 m.$$

Also
$$\frac{dW}{d\tau} = \frac{t}{t_0}\frac{dW}{dt} = \frac{t}{t_0}F.\left(V - \frac{P}{t}\right) = \Phi.\frac{d\Pi}{d\tau} = \frac{d\Omega}{d\tau}, \qquad (47)$$

so that the rate of increase of the kinetic energy is equal to the rate of performance of work by the external force in pushing the particle relative to the now stationary substratum.

98. Angular momentum in t- and τ-measure.

Angular momentum is particularly interesting in both t-measure and τ-measure.

The angular momentum of a particle at P about the origin is $H(O)$, given by

$$H(O) = m\mathbf{P} \wedge \mathbf{V}.$$

This becomes in τ-measure

$$H(O) = m\left(\frac{t}{t_0}\right)\mathbf{\Pi} \wedge \left(\frac{\mathbf{\Pi}}{t_0} + \frac{d\mathbf{\Pi}}{d\tau}\right)$$

$$= \frac{t}{t_0} m\mathbf{\Pi} \wedge \frac{d\mathbf{\Pi}}{d\tau}$$

$$= \frac{t}{t_0} h(O), \tag{48}$$

where $h(O)$, given by $\quad h(O) = m\mathbf{\Pi} \wedge \dfrac{d\mathbf{\Pi}}{d\tau},$

is the angular momentum on the τ-scale. Now if the external force $\mathbf{F}$ passes through the origin, so that $\mathbf{P} \wedge \mathbf{F} = 0$, the approximate rate of increase of angular momentum in t-measure is

$$\frac{dH(O)}{dt} = m\left(\mathbf{P} \wedge \frac{d\mathbf{V}}{dt}\right) = -m\left(\mathbf{P} \wedge \frac{\mathbf{P} - \mathbf{V}t}{t^2}\right)$$

$$= m\frac{(\mathbf{P} \wedge \mathbf{V})}{t}$$

$$= H(O)/t. \tag{49}$$

The integral of this is $\quad H(O) = \mathbf{A}t, \tag{50}$

where $\mathbf{A}$ is a vector constant. This gives a secular increase of angular momentum in time in t-measure. In τ-measure, (48) now gives

$$h(O) = \mathbf{A}t_0,$$

so that angular momentum is constant in τ-measure. Numerically the value of the angular momentum in τ-measure is equal to its value in t-measure at the instant $t = t_0$ at which the t- and τ-clocks agree.

99. Secular increase of angular momentum in the world at large. Nevertheless, the actual measure of an angular momentum on the τ-scale will secularly increase as the observer using the τ-graduated clock adjusts his value of t_0 from time to time to keep his clock in momentary agreement with the t-clock. In his *calculations* the τ-observer will regard angular momentum as constant, but the value of the constant will need to be readjusted as the value appropriate for t_0 advances. This seems to me to explain the origin of

angular momentum in the universe at large. Any system which is exposed to zero moment of the external forces will augment its stock of angular momentum as time advances. It may be supposed to have been initially (i.e. near $t = 0$) endowed with some random amount of angular momentum; this then goes on increasing proportionally to the time reckoned from the natural time-origin, owing to the couple exerted by the pull of the rest of the substratum in t-measure.

100. Exact form of angular-momentum integral. The exact form, to which (49) is an approximation, is readily obtained as follows. Multiply the equation of motion (4) vectorially by $\mathbf{P}$, and suppose $\mathbf{P} \wedge \mathbf{F} = 0$. Then we get

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{\mathbf{P} \wedge \mathbf{V}}{Y^{\frac{1}{2}}} \right) = + \frac{m\xi^{\frac{1}{2}}}{X}(\mathbf{P} \wedge \mathbf{V})\frac{Z}{Y},$$

or

$$\frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{\mathbf{P} \wedge \mathbf{V}}{Y^{\frac{1}{2}}} \right) - m\xi^{\frac{1}{2}} \frac{\mathbf{P} \wedge \mathbf{V}}{Y^{\frac{1}{2}}} \frac{Z}{X} = 0. \tag{51}$$

But

$$\frac{Z}{X} = \frac{d}{dt}\log X^{\frac{1}{2}}.$$

Hence the integrating factor of (51), which is

$$\exp\left(- \int \frac{Z}{X} dt \right),$$

reduces to $X^{-\frac{1}{2}}$.

Hence (51) may be written

$$\frac{d}{dt}\left(m\xi^{\frac{1}{2}} \frac{\mathbf{P} \wedge \mathbf{V}}{Y^{\frac{1}{2}}} \frac{1}{X^{\frac{1}{2}}} \right) = 0,$$

whence

$$m\xi^{\frac{1}{2}} \frac{\mathbf{P} \wedge \mathbf{V}}{Y^{\frac{1}{2}}} = AX^{\frac{1}{2}}. \tag{52}$$

In the case of a *free* particle, since then $\xi^{\frac{1}{2}} = $ const., the integral of angular momentum reduces to

$$\mathbf{P} \wedge \mathbf{V} = \text{const. } Z. \tag{53}$$

VII

LAGRANGIAN DEVELOPMENTS

101. WE proceed now to transform the general equations of motion obtained in the last chapter into τ-measure. To do this it is convenient first to put them in Lagrangian form. We obtained the Lagrangian form of the equations of motion of a *free* particle direct from a variational principle. But no variational principle suggests itself for the motion of a constrained particle.

102. Equations of motion in Lagrangian form. The equations of motion

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(\mathbf{P}-\mathbf{V}\frac{Z}{Y}\right)+\mathbf{F}, \tag{1}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{c}{Y^{\frac{1}{2}}}\right) = -\frac{m\xi^{\frac{1}{2}}}{X}\left(ct-c\frac{Z}{Y}\right)+F_t, \tag{1'}$$

may be rewritten in the form

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{p}}{dt} = \frac{\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}}\mathbf{p}_r+\mathbf{F}, \tag{2}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{dp_t}{dt} = \frac{\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}}(p_r)_t+F_t, \tag{2'}$$

where

$$\mathbf{p} = m\xi^{\frac{1}{2}}\frac{\mathbf{V}}{Y^{\frac{1}{2}}}, \qquad p_t = m\xi^{\frac{1}{2}}\frac{c}{Y^{\frac{1}{2}}},$$

and

$$\mathbf{p}_r = Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}} = m\xi^{\frac{1}{2}}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}-\mathbf{P}\frac{Y^{\frac{1}{2}}}{Z}\right),$$

$$(p_r)_t = -Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial c} = m\xi^{\frac{1}{2}}\left(\frac{c}{Y^{\frac{1}{2}}}-ct\frac{Y^{\frac{1}{2}}}{Z}\right).$$

We repeat also identity (37) of the last chapter,

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right) \equiv -\frac{m\xi^{\frac{1}{2}}}{X}\left(\mathbf{P}-\mathbf{V}\frac{X}{Z}\right)+\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{m\xi^{\frac{1}{2}}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}-\mathbf{P}\frac{Y^{\frac{1}{2}}}{Z}\right)\right\}, \tag{3}$$

and the analogous identity in the time-components

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(m\xi^{\frac{1}{2}}\frac{c}{Y^{\frac{1}{2}}}\right) \equiv -\frac{m\xi^{\frac{1}{2}}}{X}\left(ct-c\frac{X}{Z}\right)+\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{m\xi^{\frac{1}{2}}\left(\frac{c}{Y^{\frac{1}{2}}}-ct\frac{Y^{\frac{1}{2}}}{Z}\right)\right\}. \tag{3'}$$

These identities may be rewritten in the form

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{p}}{dt} \equiv -\frac{\partial\Omega}{\partial\mathbf{P}} + \frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{p}_r}{dt}, \tag{4}$$

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d(p)_t}{dt} \equiv +\frac{\partial\Omega}{c\partial t} + \frac{1}{Y^{\frac{1}{2}}}\frac{d(p_r)_t}{dt}. \tag{4'}$$

Now eliminate $d\mathbf{p}/dt$ between equation (2) and identity (4). We get

$$\mathbf{F} + \frac{\partial\Omega}{\partial\mathbf{P}} = \frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{p}_r}{dt} - \frac{\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}}\mathbf{p}_r = \frac{1}{Y^{\frac{1}{2}}}\left(\frac{d\mathbf{p}_r}{dt} - \frac{Z}{X}\mathbf{p}_r\right),$$

or, since $dX/dt = 2Z$,

$$\mathbf{F} + \frac{\partial\Omega}{\partial\mathbf{P}} = \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{p}_r}{X^{\frac{1}{2}}}\right), \tag{6}$$

or, again,

$$\mathbf{F} + \frac{\partial\Omega}{\partial\mathbf{P}} = \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\frac{\partial\Omega}{\partial\mathbf{V}}\right). $$

Similarly for the time-component,

$$F_t - \frac{\partial\Omega}{c\partial t} = \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\left(-\frac{\partial\Omega}{\partial c}\right)\right\}. \tag{6'}$$

(Here $\partial\Omega/\partial c$ is supposed obtained by writing Ω in the form

$$\Omega = mc^2 \frac{c.ct - \mathbf{V}.\mathbf{P}}{(c^2t^2 - \mathbf{P}^2)^{\frac{1}{2}}(c^2 - \mathbf{V}^2)^{\frac{1}{2}}}, \tag{7}$$

putting P_t for ct and then differentiating partially with respect to c where it plays a part analogous to $\mathbf{V}$, i.e. in the numerator and in $(c^2 - \mathbf{V}^2)^{-\frac{1}{2}}$.)

Formulae (6), (6'), since Ω represents kinetic energy, exhibit the equations of motion, in Lagrangian form.

103. The energy-integrals re-derived. We pause for a moment to derive the energy-integrals directly from (6) and (6'). Multiplying (6) scalarly by $\mathbf{V}/Y^{\frac{1}{2}}$, multiplying (6') by $c/Y^{\frac{1}{2}}$ and subtracting, we get

$$\frac{1}{Y^{\frac{1}{2}}}\left(\frac{\partial\Omega}{\partial t} + \mathbf{V}.\frac{\partial\Omega}{\partial\mathbf{P}}\right) + \left(\mathbf{F}.\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - F_t\frac{c}{Y^{\frac{1}{2}}}\right)$$

$$= \frac{1}{Y^{\frac{1}{2}}}\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\left(\mathbf{V}.\frac{\partial\Omega}{\partial\mathbf{V}} + c\frac{\partial\Omega}{\partial c}\right)\right\} - \frac{1}{Y^{\frac{1}{2}}}\left(\dot{\mathbf{V}}.\frac{\partial\Omega}{\partial\mathbf{V}}\right). \tag{8}$$

Now in the sense in which we use $\partial\Omega/\partial c$, since by (7) Ω is a homo-

geneous function of V and c of degree zero, we must have, by Euler's theorem on homogeneous functions,

$$V \cdot \frac{\partial \Omega}{\partial V} + c \frac{\partial \Omega}{\partial c} = 0. \tag{9}$$

Also

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial t} + V \cdot \frac{\partial \Omega}{\partial P} + \dot{V} \cdot \frac{\partial \Omega}{\partial V}. \tag{10}$$

Hence (8) may be written

$$F_t \frac{c}{Y^{\frac{1}{2}}} - F \cdot \frac{V}{Y^{\frac{1}{2}}} = \frac{1}{Y^{\frac{1}{2}}} \frac{d\Omega}{dt}. \tag{11}$$

This, the pseudo-energy integral, is as we have seen an expression of the identical relation which exists between (6) and (6′).

104. To obtain the true energy integral, multiply (6) scalarly by **P** and (6′) by ct and subtract. We get

$$\left(P \cdot \frac{\partial \Omega}{\partial P} + t \frac{\partial \Omega}{\partial t} \right) + (F \cdot P - F_t ct)$$

$$= \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}} \frac{d}{dt} \left\{ \frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} \left(P \cdot \frac{\partial \Omega}{\partial V} + ct \frac{\partial \Omega}{\partial c} \right) \right\} - \left(V \cdot \frac{\partial \Omega}{\partial V} + c \frac{\partial \Omega}{\partial c} \right).$$

The first term on the left-hand side and the second term on the right-hand side vanish by Euler's theorem on homogeneous functions. Also

$$\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} \left(P \cdot \frac{\partial \Omega}{\partial V} + ct \frac{\partial \Omega}{\partial c} \right) = \frac{m\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}} \left\{ P \cdot \left(\frac{V}{Y^{\frac{1}{2}}} - P \frac{Y^{\frac{1}{2}}}{Z} \right) - ct \left(\frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right) \right\}$$

$$= m\xi^{\frac{1}{2}} \left(\frac{1}{\xi^{\frac{1}{2}}} - \xi^{\frac{1}{2}} \right) c^2 = -mc^2(\xi - 1).$$

Hence we get

$$F \cdot P - F_t ct = -\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}} 2\xi^{\frac{1}{2}} \frac{d}{dt} (mc^2 \xi^{\frac{1}{2}}),$$

or

$$\frac{Y^{\frac{1}{2}}}{Z} (F \cdot P - F_t ct) = -\frac{2}{Y^{\frac{1}{2}}} \frac{d\Omega}{dt}.$$

Adding this to (11), we get

$$F \cdot \left(\frac{V}{Y^{\frac{1}{2}}} - P \frac{Y^{\frac{1}{2}}}{Z} \right) - F_t \left(\frac{c}{Y^{\frac{1}{2}}} - ct \frac{Y^{\frac{1}{2}}}{Z} \right) = \frac{1}{Y^{\frac{1}{2}}} \frac{d\Omega}{dt}, \tag{12}$$

which is the true energy-integral.

105. Expression of external force in terms of a potential function. We now seek as before to express F in terms of a potential

function χ, a function of P and ct such that χ is of dimensions of an energy, and accordingly such that χ/mc^2 is of zero dimensions. We find as before that to put

$$\mathbf{F} = -\frac{\partial \chi}{\partial \mathbf{P}}, \qquad F_t = +\frac{\partial \chi}{c \partial t},$$

leads to a contradiction. If therefore we put

$$\mathbf{F} = -\frac{\partial \chi}{\partial \mathbf{P}} + \alpha \mathbf{P} + \beta \frac{\mathbf{V}}{Y^{\frac{1}{2}}}, \qquad F_t = \frac{\partial \chi}{c \partial t} + \alpha ct + \beta \frac{c}{Y^{\frac{1}{2}}}, \qquad (13)$$

and if we also require the energy-integrals to reduce to

$$\chi + \Omega = \text{const.},$$

we find without trouble that we must take

$$\alpha = 0, \qquad \beta = \frac{2}{Y^{\frac{1}{2}}} \frac{d}{dt} m \xi^{\frac{1}{2}}. \qquad (14)$$

Hence the equations of motion of a particle under a potential χ are of the form

$$\frac{\partial(\Omega - \chi)}{\partial \mathbf{P}} + 2 \frac{\mathbf{V}}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d(m\xi^{\frac{1}{2}})}{dt} = \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}} \frac{d}{dt} \left(\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} \frac{\partial \Omega}{\partial \mathbf{V}} \right), \qquad (15)$$

$$-\frac{\partial(\Omega - \chi)}{c \partial t} + 2 \frac{c}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d(m\xi^{\frac{1}{2}})}{dt} = \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}} \frac{d}{dt} \left\{ \frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}} \left(-\frac{\partial \Omega}{\partial c} \right) \right\}. \qquad (15')$$

As a verification, multiplying (15) scalarly by $\mathbf{V}$ and (15') by c, we get on subtracting

$$\frac{d(\Omega - \chi)}{dt} - 2 \frac{d\Omega}{dt} = 0,$$

or

$$\frac{d}{dt}(\Omega + \chi) = 0, \qquad (16)$$

showing that χ is the actual potential energy.

106. General transformation from t-measure to τ-measure.

The transformation from t-measure to τ-measure takes the form, as we have seen (Chapters II, III, IV),

$$\mathbf{P} = ct_0 \, \mathbf{i} e^{(\tau - t_0)/t_0} \sinh \Pi/ct_0, \qquad (17)$$

$$ct = ct_0 \, e^{(\tau - t_0)/t_0} \cosh \Pi/ct_0, \qquad (17')$$

where $\mathbf{i}$ is a unit vector. We may replace the first (vector) of those equations by three equations of transformation by putting

$$\mathbf{i} = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi).$$

But it seems preferable to make the transformation of coordinates

not merely from $(t, \mathbf{P})$ to $(\tau, \mathbf{\Pi})$, but directly from $(t, \mathbf{P})$ to any set of coordinates obtained by a second space-transformation of the coordinates corresponding to $\mathbf{\Pi}$.

107. With this object, let us attempt to transform the equations of motion (15), (15′) from coordinates $(\mathbf{P}, t)$ to any permissible coordinates (q_1, q_2, q_3, τ), transforming the time-scale from t to τ. We write accordingly

$$\mathbf{P} = ct_0\, e^{(\tau - t_0)/t_0}\mathbf{f}(q_1, q_2, q_3), \tag{18}$$

$$ct = ct_0\, e^{(\tau - t_0)/t_0}f_0(q_1, q_2, q_3), \tag{18'}$$

where $\mathbf{f}$ is a vector function of three scalar coordinates q_1, q_2, q_3, f_0 a scalar function of the same three coordinates. Then[†]

$$ds^2 = dt^2 - d\mathbf{P}^2/c^2 \tag{19}$$

$$= t_0^2\, e^{2(\tau - t_0)/t_0}\left\{ \frac{f_0^2 - \mathbf{f}^2}{t_0^2}d\tau^2 + \left(\frac{f_0}{t_0}\frac{\partial f_0}{\partial q_s} - \frac{\mathbf{f}}{t_0}\cdot\frac{\partial \mathbf{f}}{\partial q_s}\right)d\tau\, dq_s - \right.$$
$$\left. - \left(\frac{\partial \mathbf{f}}{\partial q_r}\cdot\frac{\partial \mathbf{f}}{\partial q_s} - \frac{\partial f_0}{\partial q_r}\frac{\partial f_0}{\partial q_s}\right)dq_r\, dq_s \right\}. \tag{20}$$

We shall choose f_0, $\mathbf{f}$ so that ds reduces to the form

$$ds = \Delta e^{(\tau - t_0)/t_0}d\sigma, \tag{21}$$

where

$$\Delta = (f_0^2 - \mathbf{f}^2)^{\frac{1}{2}}, \tag{22}$$

and

$$d\sigma^2 = d\tau^2 - d\epsilon^2/c^2, \tag{23}$$

where

$$d\epsilon^2 = \frac{(ct_0)^2}{\Delta^2}\left(\frac{\partial \mathbf{f}}{\partial q_r}\cdot\frac{\partial \mathbf{f}}{\partial q_s} - \frac{\partial f_0}{\partial q_r}\frac{\partial f_0}{\partial q_s}\right)dq_r\, dq_s, \tag{24}$$

and accordingly

$$f_0\frac{\partial f_0}{\partial q_s} - \mathbf{f}\cdot\frac{\partial \mathbf{f}}{\partial q_s} = 0 \quad (s = 1, 2, 3). \tag{25}$$

Conditions (25) imply $\Delta = \text{const.}$

Further, we have

$$X^{\frac{1}{2}} = (t^2 - \mathbf{P}^2/c^2)^{\frac{1}{2}} = t_0\, e^{(\tau - t_0)/t_0}\Delta, \tag{26}$$

so that

$$dX^{\frac{1}{2}} = \Delta e^{(\tau - t_0)/t_0}d\tau.$$

Also

$$Y^{\frac{1}{2}}dt = ds = \Delta e^{(\tau - t_0)/t_0}d\sigma.$$

But

$$\frac{1}{Y^{\frac{1}{2}}}\frac{dX^{\frac{1}{2}}}{dt} = \frac{Z}{X^{\frac{1}{2}}Y^{\frac{1}{2}}} = \xi^{\frac{1}{2}}.$$

Hence

$$\xi^{\frac{1}{2}} = \frac{d\tau}{d\sigma} = \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}}, \tag{27}$$

[†] We use the summation convention for repeated suffixes.

where $\qquad v^2 = \left(\dfrac{d\epsilon}{d\tau}\right)^2 = \dfrac{(ct_0)^2}{\Delta^2}\left(\dfrac{\partial \mathbf{f}}{\partial q_r}\cdot\dfrac{\partial \mathbf{f}}{\partial q_s} - \dfrac{\partial f_0}{\partial q_r}\dfrac{\partial f_0}{\partial q_s}\right)\dot{q}_r\,\dot{q}_s.$ (28)

In this, $\dot{q}_s$ stands for $dq_s/d\tau$.

108. Programme of the reduction to τ-measure.

We now recall that since the potential χ is a function of $\mathbf{P}$ and t, therefore

$$\frac{\partial \chi}{\partial q_r} = \frac{\partial \chi}{\partial \mathbf{P}}\cdot\frac{\partial \mathbf{P}}{\partial q_r} + \frac{\partial \chi}{c\,\partial t}\frac{c\,\partial t}{\partial q_r},$$ (29)

whilst on the other hand Ω is a function of $\mathbf{P}$, t, and $\mathbf{V}$, so that

$$\frac{\partial \Omega}{\partial q_r} = \frac{\partial \Omega}{\partial \mathbf{P}}\cdot\frac{\partial \mathbf{P}}{\partial q_r} + \frac{\partial \Omega}{c\,\partial t}\frac{c\,\partial t}{\partial q_r} + \frac{\partial \Omega}{\partial \mathbf{V}}\cdot\frac{\partial \mathbf{V}}{\partial q_r} + \frac{\partial \Omega}{\partial c}\frac{\partial c}{\partial q_r}.$$ (30)

In the latter equality we have added the zero term $(\partial\Omega/\partial c)(\partial c/\partial q_r)$ for symmetry's sake.

Now multiply the Lagrangian t-equations (15) and (15') by $\partial\mathbf{P}/\partial q_r$ (scalarly) and $-c\,\partial t/\partial q_r$, respectively, and add. The result may be written, using (29) and (30),

$$\left(-\frac{\partial \chi}{\partial q_r} + \frac{\partial \Omega}{\partial q_r}\right) - \left(\frac{\partial \Omega}{\partial \mathbf{V}}\cdot\frac{\partial \mathbf{V}}{\partial q_r} + \frac{\partial \Omega}{\partial c}\frac{\partial c}{\partial q_r}\right) + \frac{2}{Y}\frac{d}{dt}(m\xi^{\frac{1}{2}})\left(\mathbf{V}\cdot\frac{\partial \mathbf{P}}{\partial q_r} - c\frac{\partial(ct)}{\partial q_r}\right)$$
$$= \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\left(\frac{\partial \Omega}{\partial \mathbf{V}}\cdot\frac{\partial \mathbf{P}}{\partial q_r} + \frac{\partial \Omega}{\partial c}\frac{\partial(ct)}{\partial q_r}\right)\right\} - \left\{\frac{\partial \Omega}{\partial \mathbf{V}}\frac{d}{dt}\left(\frac{\partial \mathbf{P}}{\partial q_r}\right) + \frac{\partial \Omega}{\partial c}\frac{d}{dt}\frac{\partial(ct)}{\partial q_r}\right\}. \quad (31)$$

We shall reduce this equation by combining the third term on the left-hand side with the first term on the right-hand side; and we shall show that the second term on the left-hand side cancels the second term on the right-hand side.

109. Execution of details.

We begin by noting that, since

$$\left(\frac{d\sigma}{d\tau}\right)^2 = 1 - \frac{t_0^2}{\Delta^2}\left(\frac{\partial \mathbf{f}}{\partial q_r}\cdot\frac{\partial \mathbf{f}}{\partial q_s} - \frac{\partial f_0}{\partial q_r}\frac{\partial f_0}{\partial q_s}\right)\dot{q}_r\,\dot{q}_s,$$

therefore, differentiating partially with regard to $\dot{q}_r$,

$$\left(\frac{d\sigma}{d\tau}\right)\frac{\partial}{\partial \dot{q}_r}\left(\frac{d\sigma}{d\tau}\right) = -\frac{t_0^2}{\Delta^2}\left(\frac{\partial \mathbf{f}}{\partial q_r}\cdot\frac{\partial \mathbf{f}}{\partial q_s} - \frac{\partial f_0}{\partial q_r}\frac{\partial f_0}{\partial q_s}\right)\dot{q}_s.$$ (32)

Further

$$\frac{\mathbf{V}}{Y^{\frac{1}{2}}} = \frac{1}{Y^{\frac{1}{2}}}\frac{d\mathbf{P}}{dt} = \frac{d\mathbf{P}}{ds} = \frac{e^{-(\tau-t_0)/t_0}}{\Delta}\frac{d\mathbf{P}}{d\sigma} = \frac{e^{-(\tau-t_0)/t_0}}{\Delta}\frac{d\tau}{d\sigma}\frac{d\mathbf{P}}{d\tau},$$ (33)

and similarly $\qquad \dfrac{c}{Y^{\frac{1}{2}}} = \dfrac{e^{-(\tau-t_0)/t_0}}{\Delta}\dfrac{d\tau}{d\sigma}\dfrac{d(ct)}{d\tau}.$ (33')

Also $\quad \dfrac{1}{Y^{\frac{1}{2}}}\dfrac{d(m\xi^{\frac{1}{2}})}{dt} = m\dfrac{e^{-(\tau-t_0)/t_0}}{\Delta}\dfrac{d}{d\sigma}\left(\dfrac{d\tau}{d\sigma}\right) = m\dfrac{e^{-(\tau-t_0)/t_0}}{\Delta}\dfrac{d\tau}{d\sigma}\dfrac{d}{d\tau}\left(\dfrac{d\tau}{d\sigma}\right).$ $\qquad$ (34)

Hence the third term on the left-hand side of (31), namely

$$\dfrac{2}{Y^{\frac{1}{2}}}\dfrac{d(m\xi^{\frac{1}{2}})}{dt}\left(\dfrac{\mathbf{V}}{Y^{\frac{1}{2}}}\cdot\dfrac{\partial\mathbf{P}}{\partial q_r} - \dfrac{c}{Y^{\frac{1}{2}}}\dfrac{c}{\partial q_r}\dfrac{\partial t}{}\right), \qquad (35)$$

is equal to

$$2m\dfrac{e^{-2(\tau-t_0)/t_0}}{\Delta^2}\left(\dfrac{d\tau}{d\sigma}\right)^2\dfrac{d}{d\tau}\left(\dfrac{d\tau}{d\sigma}\right)\left(\dfrac{d\mathbf{P}}{d\tau}\cdot\dfrac{\partial\mathbf{P}}{\partial q_r} - \dfrac{d(ct)}{d\tau}\dfrac{\partial(ct)}{\partial q_r}\right).$$

Substituting from (18) and (18′) in this, it becomes

$$-\dfrac{2}{\Delta^2}m\dfrac{\dfrac{d}{d\tau}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^2}\left(\dfrac{d\tau}{d\sigma}\right)^2\left\{\left(\dfrac{\mathbf{f}}{t_0}+\dfrac{\partial\mathbf{f}}{\partial q_s}\dot{q}_s\right)\cdot\dfrac{\partial\mathbf{f}}{\partial q_r} - \left(\dfrac{f_0}{t_0}+\dfrac{\partial f_0}{\partial q_s}\dot{q}_s\right)\dfrac{\partial f_0}{\partial q_r}\right\}(ct_0)^2.$$

Using (25) and the equality (32), we have that (35) is equal to

$$2c^2m\dfrac{d}{d\tau}\left(\dfrac{d\sigma}{d\tau}\right)\left(\dfrac{d\tau}{d\sigma}\right)^4\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial\dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right) = 2c^2m\left(\dfrac{d\tau}{d\sigma}\right)^3\dfrac{d}{d\tau}\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial\dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right). \quad (36)$$

Now consider the first term on the right-hand side of (31). We have

$$\dfrac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\dfrac{d}{dt} = t_0\dfrac{d}{d\sigma} = t_0\dfrac{d\tau}{d\sigma}\dfrac{d}{d\tau}. \qquad (37)$$

Also

$$Y^{\frac{1}{2}}\dfrac{\partial\Omega}{\partial\mathbf{V}} = m\xi^{\frac{1}{2}}\left(\dfrac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P}\dfrac{Y^{\frac{1}{2}}}{Z}\right) = m\dfrac{d\tau}{d\sigma}\left(\dfrac{e^{-(\tau-t_0)/t_0}}{\Delta}\dfrac{d\mathbf{P}}{d\tau}\dfrac{d\tau}{d\sigma}\right) - m\dfrac{\mathbf{P}}{X^{\frac{1}{2}}},$$

and $\qquad\qquad\qquad\qquad m\dfrac{\mathbf{P}}{X^{\frac{1}{2}}} = \dfrac{mc}{\Delta}\mathbf{f}.$

Hence

$$\dfrac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\dfrac{\partial\Omega}{\partial\mathbf{V}}\cdot\dfrac{\partial\mathbf{P}}{\partial q_r} = m\dfrac{e^{-(\tau-t_0)/t_0}}{\Delta}\left(\dfrac{d\tau}{d\sigma}\right)^2 e^{(\tau-t_0)/t_0}\left(\dfrac{\mathbf{f}}{t_0}+\dfrac{\partial\mathbf{f}}{\partial q_s}\dot{q}_s\right)\times$$

$$\times\, e^{(\tau-t_0)/t_0}\dfrac{\partial\mathbf{f}}{\partial q_r}\dfrac{(ct_0)^2}{t_0\Delta e^{(\tau-t_0)/t_0}} - \dfrac{mc}{\Delta}\mathbf{f}\cdot\dfrac{e^{(\tau-t_0)/t_0}}{\Delta e^{(\tau-t_0)/t_0}t_0}\dfrac{\partial\mathbf{f}}{\partial q_r}.$$

Subtracting the corresponding terms in f_0 and again using (32), we get

$$\dfrac{mc^2 t_0}{\Delta^2}\left(\dfrac{d\tau}{d\sigma}\right)^2\left(\dfrac{\partial\mathbf{f}}{\partial q_r}\cdot\dfrac{\partial\mathbf{f}}{\partial q_s} - \dfrac{\partial f_0}{\partial q_r}\dfrac{\partial f_0}{\partial q_s}\right)\dot{q}_s = -\dfrac{mc^2}{t_0}\left(\dfrac{d\tau}{d\sigma}\right)^2\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial\dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)$$

$$= -\dfrac{mc^2}{t_0}\left(\dfrac{d\tau}{d\sigma}\right)\dfrac{\partial}{\partial\dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right).$$

Acting on this with the operator (37), we find that the first term on the right-hand side of (31) comes to

$$-\left(\frac{d\tau}{d\sigma}\right)\frac{d}{d\tau}\left\{mc^2\left(\frac{d\tau}{d\sigma}\right)\frac{\partial}{\partial \dot{q}_r}\left(\frac{d\sigma}{d\tau}\right)\right\} = -mc^2\left(\frac{d\tau}{d\sigma}\right)\frac{d}{d\tau}\left\{\frac{\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\dfrac{d\sigma}{d\tau}}\right\}$$

$$= mc^2\frac{\dfrac{d}{d\tau}\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^3} - mc^2\frac{\dfrac{d}{d\tau}\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^2}. \tag{38}$$

Bringing now (36) over to the right-hand side of (31) and combining it with (38), we get altogether

$$-mc^2\frac{\dfrac{d}{d\tau}\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^3} - mc^2\frac{\dfrac{d}{d\tau}\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^2}$$

$$= -mc^2\frac{\dfrac{d}{d\tau}\left\{\left(\dfrac{d\sigma}{d\tau}\right)\dfrac{\partial}{\partial \dot{q}_r}\left(\dfrac{d\sigma}{d\tau}\right)\right\}}{\left(\dfrac{d\sigma}{d\tau}\right)^3} = -mc^2\frac{\dfrac{d}{d\tau}\left\{\dfrac{\partial}{\partial \dot{q}_r}\dfrac{1}{2}\left(\dfrac{d\sigma}{d\tau}\right)^2\right\}}{\left(\dfrac{d\sigma}{d\tau}\right)^3}. \tag{39}$$

110. We now want to show that

$$-\frac{\partial\Omega}{\partial V}\cdot\frac{\partial V}{\partial q_r} - \frac{\partial\Omega}{\partial c}\frac{\partial c}{\partial q_r} = -\frac{\partial\Omega}{\partial V}\cdot\frac{d}{dt}\frac{\partial P}{\partial q_r} - \frac{\partial\Omega}{\partial c}\frac{d}{dt}\frac{\partial(ct)}{\partial q_r}. \tag{40}$$

Consider the right-hand side of (40). It may be written

$$-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial V}\cdot\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\frac{\partial P}{\partial q_r} - Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial c}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\frac{\partial(ct)}{\partial q_r}$$

$$= -Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial V}\cdot\frac{e^{-(\tau-t_0)/t_0}}{\Delta}\frac{d\tau}{d\sigma}\frac{d}{d\tau}\frac{\partial P}{\partial q_r} - \dots. \tag{41}$$

But

$$\frac{d}{d\tau}\frac{\partial P}{\partial q_r} = \frac{d}{d\tau}\left(ct_0\,e^{(\tau-t_0)/t_0}\frac{\partial f}{\partial q_r}\right)$$

$$= ct_0\,e^{(\tau-t_0)/t_0}\left(\frac{1}{t_0}\frac{\partial f}{\partial q_r} + \frac{\partial^2 f}{\partial q_r\,\partial q_s}\dot{q}_s\right)$$

$$= ct_0\frac{\partial}{\partial q_r}\left\{e^{(\tau-t_0)/t_0}\left(\frac{f}{t_0} + \frac{\partial f}{\partial q_s}\dot{q}_s\right)\right\}$$

$$= \frac{\partial}{\partial q_r}\frac{dP}{d\tau}. \tag{42}$$

Now consider the left-hand side of (40). It may be written

$$-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{1}{Y^{\frac{1}{2}}}\frac{\partial\mathbf{V}}{\partial q_r}-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial c}\frac{1}{Y^{\frac{1}{2}}}\frac{\partial c}{\partial q_r}.$$

In this, the first term may be written

$$-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial}{\partial q_r}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right)+Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\mathbf{V}\frac{\partial}{\partial q_r}\left(\frac{1}{Y^{\frac{1}{2}}}\right).$$

But

$$\mathbf{V}\cdot\frac{\partial\Omega}{\partial\mathbf{V}}+c\frac{\partial\Omega}{\partial c}=0, \tag{43}$$

and

$$-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial}{\partial q_r}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right)-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial c}\frac{\partial}{\partial q_r}\left(\frac{c}{Y^{\frac{1}{2}}}\right)$$

$$=-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial}{\partial q_r}\left(\frac{e^{-(\tau-l_0)/l_0}}{\Delta}\frac{d\mathbf{P}}{d\tau}\frac{d\tau}{d\sigma}\right)-\cdots$$

$$=-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{e^{-(\tau-l_0)/l_0}}{\Delta}\frac{d\tau}{d\sigma}\frac{\partial}{\partial q_r}\left(\frac{d\mathbf{P}}{d\tau}\right)-\cdots$$

$$-Y^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{e^{-(\tau-l_0)/l_0}}{\Delta}\frac{d\mathbf{P}}{d\tau}\frac{\partial}{\partial q_r}\left(\frac{d\tau}{d\sigma}\right)-\cdots. \tag{44}$$

The first terms on the right-hand side of (44) are cancelled by (41), on using (42); and the remaining terms in (44) give

$$-Y^{\frac{1}{2}}\frac{e^{-(\tau-l_0)/l_0}}{\Delta}\frac{\partial}{\partial q_r}\left(\frac{d\tau}{d\sigma}\right)\left[\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{d\mathbf{P}}{d\tau}+\frac{\partial\Omega}{\partial c}\frac{d(ct)}{d\tau}\right].$$

Here the square bracket may be written

$$\left[\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{d\mathbf{P}}{dt}+\frac{\partial\Omega}{\partial c}\frac{d(ct)}{dt}\right]\frac{dt}{d\tau}$$

$$=\left[\mathbf{V}\cdot\frac{\partial\Omega}{\partial\mathbf{V}}+c\frac{\partial\Omega}{\partial c}\right]\frac{dt}{d\tau}, \tag{45}$$

and this vanishes.

111. Lagrangian form of equations of motion. Returning now to (31) and using (39), we find

$$-\frac{\partial\chi}{\partial q_r}+\frac{\partial}{\partial q_r}\left(mc^2\frac{d\tau}{d\sigma}\right)=-mc^2\frac{\dfrac{d}{d\tau}\left\{\dfrac{\partial}{\partial\dot{q}_r}\dfrac{1}{2}\left(\dfrac{d\sigma}{d\tau}\right)^2\right\}}{\left(\dfrac{d\sigma}{d\tau}\right)^3}.$$

But

$$\frac{\partial}{\partial q_r}\left(\frac{d\tau}{d\sigma}\right)=-\frac{\dfrac{\partial}{\partial q_r}\left(\dfrac{d\sigma}{d\tau}\right)}{\left(\dfrac{d\sigma}{d\tau}\right)^2}=-\frac{\dfrac{\partial}{\partial q_r}\left\{\dfrac{1}{2}\left(\dfrac{d\sigma}{d\tau}\right)^2\right\}}{\left(\dfrac{d\sigma}{d\tau}\right)^3}.$$

Hence the result is

$$-\frac{\partial \chi}{\partial q_r} = -\frac{mc^2}{(d\sigma/d\tau)^3}\left\{\frac{d}{d\tau}\frac{\partial}{\partial \dot{q}_r}\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2 - \frac{\partial}{\partial q_r}\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right\}, \qquad (46)$$

or, using

$$\left(\frac{d\sigma}{d\tau}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{d\epsilon}{d\tau}\right)^2 = 1 - \frac{v^2}{c^2}, \qquad (47)$$

we have

$$-\frac{\partial \chi}{\partial q_r} = \frac{m}{(1-v^2/c^2)^{\frac{1}{2}}}\left(\frac{d}{d\tau}\frac{\partial}{\partial \dot{q}_r}\tfrac{1}{2}v^2 - \frac{\partial}{\partial q_r}\tfrac{1}{2}v^2\right). \qquad (48)$$

This is the equation of motion, for the variable q_r, for a particle moving under an external field of potential χ, expressed in τ-measure. It is in what we may call Lagrangian form, since it reduces strictly to the standard dynamical equation of Lagrange when $v \ll c$. This again verifies that τ is the independent time-variable for the dynamics of Newton-Lagrange.

112. Energy-integral from Lagrangian equations.

As a verification, we proceed to derive the energy-integral in τ-measure directly from equations (48), or the equivalent form (46). Multiplying (46) by $\dot{q}_r$ and carrying out the summation implied by repetition of the suffix r, we have

$$-\dot{q}_r\frac{\partial \chi}{\partial q_r}$$

$$= -\frac{mc^2}{(d\sigma/d\tau)^3}\left[\frac{d}{d\tau}\left\{\dot{q}_r\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right)\right\} - \dot{q}_r\frac{\partial}{\partial q_r}\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2 - \ddot{q}_r\frac{\partial}{\partial \dot{q}_r}\left\{\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right\}\right]. \qquad (49)$$

But

$$\left(\dot{q}_r\frac{\partial}{\partial q_r} + \ddot{q}_r\frac{\partial}{\partial \dot{q}_r}\right)\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2 = \frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right\}.$$

And if we write

$$\left(\frac{d\sigma}{d\tau}\right)^2 = 1 - \frac{c^2 t_0^2}{c^2}g_{rs}\dot{q}_r\dot{q}_s,$$

then

$$\dot{q}_r\frac{\partial}{\partial \dot{q}_r}\left(\frac{d\sigma}{d\tau}\right)^2 = -2\frac{c^2 t_0^2}{c^2}g_{rs}\dot{q}_s\dot{q}_r = 2\left\{\left(\frac{d\sigma}{d\tau}\right)^2 - 1\right\}. \qquad (50)$$

Combining these results,

$$-\dot{q}_r\frac{\partial \chi}{\partial q_r} = -\frac{mc^2}{(d\sigma/d\tau)^3}\left[\frac{d}{d\tau}\left\{\left(\frac{d\sigma}{d\tau}\right)^2 - 1\right\} - \frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right\}\right]$$

$$= -\frac{mc^2}{(d\sigma/d\tau)^3}\frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2\right\} = -\frac{mc^2}{(d\sigma/d\tau)^2}\frac{d}{d\tau}\left(\frac{d\sigma}{d\tau}\right),$$

or

$$-\frac{d\chi}{d\tau} = +mc^2\frac{d}{d\tau}\frac{1}{d\sigma/d\tau} = mc^2\frac{d}{d\tau}\left(\frac{d\tau}{d\sigma}\right). \qquad (51)$$

This integrates as it stands in the form

$$\frac{\chi}{mc^2} + \frac{d\tau}{d\sigma} = \text{const.,}$$

or

$$\chi + \frac{mc^2}{(1-v^2/c^2)^{\frac{1}{2}}} = \text{const.,} \qquad (52)$$

which is the energy-integral.

113. Fourth-component relation. Equations (48) have been derived from the pair of original t-equations (6) and (6') by operating with the q_r-derivative of the 4-vector $(\mathbf{P}, ct)$. It may be asked what happens if we operate with the τ-derivative of this vector.

Multiply (6) and (6') in turn by $\partial \mathbf{P}/\partial\tau$ scalarly and by $-\partial(ct)/\partial\tau$. Then since

$$\frac{\partial\Omega}{\partial\tau} = \frac{\partial\Omega}{\partial\mathbf{P}}\cdot\frac{\partial\mathbf{P}}{\partial\tau} + \frac{\partial\Omega}{c\,\partial t}\frac{c\,\partial t}{\partial\tau} + \frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial\mathbf{V}}{\partial\tau} + \frac{\partial\Omega}{\partial c}\frac{\partial c}{\partial\tau},$$

and since also

$$\frac{\partial\chi}{\partial\tau} = \frac{\partial\chi}{\partial\mathbf{P}}\cdot\frac{\partial\mathbf{P}}{\partial\tau} + \frac{\partial\chi}{c\,\partial t}\frac{\partial(ct)}{\partial\tau},$$

we get

$$\left(-\frac{\partial\chi}{\partial\tau} + \frac{\partial\Omega}{\partial\tau}\right) - \left(\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial\mathbf{V}}{\partial\tau} + \frac{\partial\Omega}{\partial c}\frac{\partial c}{\partial\tau}\right) + \frac{2}{Y}\frac{d}{dt}(m\xi^{\frac{1}{2}})\left(\mathbf{V}\cdot\frac{\partial\mathbf{P}}{\partial\tau} - c\frac{\partial(ct)}{\partial\tau}\right)$$

$$= \frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left[\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}}\left(\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{\partial\mathbf{P}}{\partial\tau} + \frac{\partial\Omega}{\partial c}\frac{\partial(ct)}{\partial\tau}\right)\right] - \left[\frac{\partial\Omega}{\partial\mathbf{V}}\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{P}}{\partial\tau}\right) + \frac{\partial\Omega}{\partial c}\frac{d}{dt}\frac{\partial(ct)}{\partial\tau}\right]. \quad (53)$$

But

$$\frac{\partial\mathbf{P}}{\partial\tau} = \frac{\mathbf{P}}{t_0}, \qquad \frac{\partial(ct)}{\partial\tau} = \frac{ct}{t_0}.$$

The contents of the first square bracket on the right-hand side accordingly come to

$$\frac{Y^{\frac{1}{2}}}{X^{\frac{1}{2}}t_0}\left(\mathbf{P}\cdot\frac{\partial\Omega}{\partial\mathbf{V}} + ct\frac{\partial\Omega}{\partial c}\right) = \frac{m\xi^{\frac{1}{2}}}{X^{\frac{1}{2}}t_0}\left\{\mathbf{P}\cdot\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P}\frac{Y^{\frac{1}{2}}}{Z}\right) - ct\left(\frac{c}{Y^{\frac{1}{2}}} - ct\frac{Y^{\frac{1}{2}}}{Z}\right)\right\}$$

$$= \frac{m\xi^{\frac{1}{2}}c^2}{t_0 X^{\frac{1}{2}}}\left\{-\frac{Z}{Y^{\frac{1}{2}}} + \frac{XY^{\frac{1}{2}}}{Z}\right\} = \frac{mc^2}{t_0}(1-\xi).$$

Hence the first term on the right-hand side comes to

$$\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left\{\frac{mc^2}{t_0}(1-\xi)\right\} = -\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{mc^2}{t_0}2\xi^{\frac{1}{2}}\frac{d\xi^{\frac{1}{2}}}{dt} = -2\frac{Z}{Y}\frac{1}{t_0}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}).$$

But the last term on the left-hand side of (53) comes to

$$\frac{2}{Y}\frac{d(m\xi^{\frac{1}{2}})}{dt}\left(\frac{\mathbf{V}.\mathbf{P}}{t_0} - \frac{c^2 t}{t_0}\right) = -2\frac{Z}{Y}\frac{1}{t_0}\frac{d(mc^2\xi^2)}{dt}.$$

These terms accordingly cancel. Similarly, by analysis similar to

that used in §110, the second term on the right-hand side of (53) cancels the second term on the left-hand side. Consequently (53) reduces to

$$\frac{\partial \Omega}{\partial \tau} - \frac{\partial \chi}{\partial \tau} = 0. \tag{54}$$

But, since Ω/mc^2 is homogeneous and of degree zero in $\mathbf{P}$ and t, in view of the way τ occurs in the transformation formulae (18), (18') it follows that $\partial \Omega/\partial \tau \equiv 0$. Hence

$$\frac{\partial \chi}{\partial \tau} = 0, \tag{55}$$

or χ must be explicitly independent of τ. This therefore is the form which the τ-Lagrangian equation takes.

114. Alternative forms of Lagrangian equations. The Lagrangian equations (46) or (48) may be expressed in a more elegant way as follows:

We have $\qquad \Omega = mc^2 \xi^{\frac{1}{2}} = mc^2 \dfrac{d\tau}{d\sigma} = \dfrac{mc^2}{(1 - v^2/c^2)^{\frac{1}{2}}}.$

Hence $\qquad\qquad\qquad \dfrac{d\sigma}{d\tau} = \dfrac{mc^2}{\Omega}. \tag{56}$

Hence the Lagrangian equation (46) in the form

$$-\frac{\partial \chi}{\partial q_r} = -\frac{mc^2}{(d\sigma/d\tau)^3}\left\{\frac{d}{d\tau}\left(\frac{d\sigma}{d\tau}\frac{\partial}{\partial \dot{q}_r}\frac{d\sigma}{d\tau}\right) - \frac{d\sigma}{d\tau}\frac{\partial}{\partial q_r}\frac{d\sigma}{d\tau}\right\}$$

may be written as

$$-\frac{\partial \chi}{\partial q_r} = -\Omega^3\left\{\frac{d}{d\tau}\left(-\frac{1}{\Omega}\frac{1}{\Omega^2}\frac{\partial \Omega}{\partial \dot{q}_r}\right) - \frac{1}{\Omega}\left(-\frac{1}{\Omega^2}\frac{\partial \Omega}{\partial q_r}\right)\right\} = \Omega^3\frac{d}{d\tau}\left(\frac{1}{\Omega^3}\frac{\partial \Omega}{\partial \dot{q}_r}\right) - \frac{\partial \Omega}{\partial q_r},$$

or $\qquad\qquad \dfrac{1}{\Omega^3}\dfrac{\partial}{\partial q_r}(\Omega - \chi) = \dfrac{d}{d\tau}\left(\dfrac{1}{\Omega^3}\dfrac{\partial \Omega}{\partial \dot{q}_r}\right). \tag{57}$

Hence if we put $\qquad\qquad L = \Omega - \chi,$

since χ does not involve $\dot{q}_r$ the last equation may be written

$$\frac{1}{\Omega^3}\frac{\partial L}{\partial q_r} = \frac{d}{d\tau}\left(\frac{1}{\Omega^3}\frac{\partial L}{\partial \dot{q}_r}\right). \tag{58}$$

115. It is instructive to see how this form yields the energy-integral. Multiply by $\dot{q}_r$ and carry out the implied summation. We get

$$\frac{1}{\Omega^3}\left(\dot{q}_r\frac{\partial \Omega}{\partial q_r} - \dot{q}_r\frac{\partial \chi}{\partial q_r}\right) = \frac{d}{d\tau}\left(\frac{\dot{q}_r}{\Omega^3}\frac{\partial \Omega}{\partial \dot{q}_r}\right) - \frac{\ddot{q}_r}{\Omega^3}\frac{\partial \Omega}{\partial \dot{q}_r},$$

or $\qquad\qquad \dfrac{1}{\Omega^3}\left(\dfrac{d\Omega}{d\tau} - \dfrac{d\chi}{d\tau}\right) = \dfrac{d}{d\tau}\left(\dfrac{\dot{q}_r}{\Omega^3}\dfrac{\partial \Omega}{\partial \dot{q}_r}\right). \tag{59}$

But by (50),
$$\dot{q}_r \frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{\Omega^2}\right) = 2\left(\frac{1}{\Omega^2} - \frac{1}{m^2 c^4}\right),$$

or
$$\frac{1}{\Omega^3}\dot{q}_r \frac{\partial \Omega}{\partial \dot{q}_r} = -\frac{1}{\Omega^2} + \frac{1}{m^2 c^4}.$$

Hence
$$\frac{d}{d\tau}\left(\frac{\dot{q}_r}{\Omega^3}\frac{\partial \Omega}{\partial \dot{q}_r}\right) = +\frac{2}{\Omega^3}\frac{d\Omega}{d\tau}.$$

Hence (59) becomes
$$\frac{1}{\Omega^3}\left(\frac{d\Omega}{d\tau} - \frac{d\chi}{d\tau}\right) = +\frac{2}{\Omega^3}\frac{d\Omega}{d\tau},$$

or
$$\frac{d(\Omega + \chi)}{d\tau} = 0,$$

or
$$\Omega + \chi = \text{const.} \tag{60}$$

which is as before the energy-integral.

116. Non-existence of a general variational principle.

In spite of its simplicity, the Lagrangian equation of motion (58) is not of the form of an Eulerian equation, and I have not found it possible to deduce a variational principle corresponding to it. When no external field of potential is present, so that we can put $\chi = 0$, equation (58), which is of course equivalent to (48), yields at once the Lagrangian equations (63) of Chapter V, which were indeed derived from the variational principles we found for the motion of a *free* particle.

117. Hamiltonian transformation.

We can pass from the exact Lagrangian equations (58) to equations of Hamiltonian form, by the usual Hamiltonian transformation, as follows. Put

$$p_r = \frac{\partial L}{\partial \dot{q}_r} = \frac{\partial \Omega}{\partial \dot{q}_r} = mc^2 \frac{\partial}{\partial \dot{q}_r}\left(\frac{d\tau}{d\sigma}\right). \tag{61}$$

This scalar p_r, which is not to be confused with the vector $\mathbf{p}_r$ of § 102 etc., is the momentum component in τ-measure corresponding to the coordinate q_r. Since

$$\frac{m^2 c^4}{\Omega^2} = \left(\frac{d\sigma}{d\tau}\right)^2 = 1 - \frac{c^2 t_0^2}{c^2} g_{rs}\dot{q}_r \dot{q}_s,$$

we have
$$-\frac{m^2 c^4}{\Omega^3}\frac{\partial \Omega}{\partial \dot{q}_r} = -\frac{c^2 t_0^2}{c^2} g_{rs}\dot{q}_s.$$

Hence $\qquad -\dfrac{m^2c^4}{\Omega^3}\,\dot{q}_r\,p_r = -\dfrac{c^2t_0^2}{c^2}\,g_{rs}\,\dot{q}_s\,\dot{q}_r = \dfrac{m^2c^4}{\Omega^2} - 1,$

or $\qquad\qquad p_r\dot{q}_r = \dfrac{\Omega^3}{m^2c^4} - \Omega.$ $\qquad\qquad$ (62)

Now put $\qquad\qquad H = p_r\dot{q}_r - L.$ $\qquad\qquad$ (63)

Then $\qquad H = \dfrac{\Omega^3}{m^2c^4} - \Omega - (\Omega - \chi) = \dfrac{\Omega^3}{m^2c^4} - 2\Omega + \chi.$ $\qquad$ (64)

Taking the variation of (63), supposing H to be expressed as a function of the q_r's and p_r's,

$$\delta H = \frac{\partial H}{\partial p_r}\delta p_r + \frac{\partial H}{\partial q_r}\delta q_r = p_r\delta\dot{q}_r + \dot{q}_r\delta p_r - \frac{\partial L}{\partial q_r}\delta q_r - \frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r,$$

or, using (61),

$$\frac{\partial H}{\partial p_r}\delta p_r + \frac{\partial H}{\partial q_r}\delta q_r = \dot{q}_r\delta p_r - \frac{\partial L}{\partial q_r}\delta q_r.$$

Equating coefficients of δp_r and δq_r,

$$\frac{\partial H}{\partial p_r} = \dot{q}_r,$$ $\qquad\qquad$ (65)

$$\frac{\partial H}{\partial q_r} = -\frac{\partial L}{\partial q_r} = -\Omega^3\frac{d}{d\tau}\left(\frac{p_r}{\Omega^3}\right),$$ $\qquad$ (66)

on using the Lagrangian equation (58). The last equation may be written

$$\frac{\partial H}{\partial q_r} = -\dot{p}_r + \frac{3}{\Omega}p_r\frac{d\Omega}{d\tau}.$$ $\qquad$ (66')

When $v \ll c$, these reduce to the Hamiltonian equations of classical mechanics.

118. Energy-integral from Hamiltonian equations.

It is again instructive to see how (65) and (66') combine to yield an energy-integral. Multiplying (65) by $\dot{p}_r$ and (66') by $\dot{q}_r$, we get on adding

$$\frac{dH}{d\tau} = \dot{p}_r\frac{\partial H}{\partial p_r} + \dot{q}_r\frac{\partial H}{\partial q_r} = \frac{3}{\Omega}\frac{d\Omega}{d\tau}p_r\dot{q}_r,$$

or, by (62),

$$\frac{dH}{d\tau} = \frac{3}{\Omega}\frac{d\Omega}{d\tau}\left(\frac{\Omega^3}{m^2c^4} - \Omega\right) = 3\frac{\Omega^2}{m^2c^4}\frac{d\Omega}{d\tau} - 3\frac{d\Omega}{d\tau}.$$

This integrates in the form

$$H = \frac{\Omega^3}{m^2c^4} - 3\Omega + \text{const.}$$ $\qquad$ (67)

Substituting for H from (64) in (67), we get at once

$$\chi + \Omega = \text{const.}$$

119. Local forms.

The hyperbolic space in which the foregoing τ-equations hold good is locally Euclidean. It is therefore of interest to see what form the equations of motion take when we choose q_1, q_2, q_3 to be local Cartesian coordinates† x, y, z but retain all powers of v/c. Then

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$
$$\Omega = mc^2/(1 - v^2/c^2)^{\frac{1}{2}}.$$

Thus v^2 is independent of the coordinates x, y, z themselves, and the Lagrangian equation in the form (48) reduces to

$$\frac{m}{(1 - v^2/c^2)^{\frac{1}{2}}} \frac{d^2x}{d\tau^2} = -\frac{\partial \chi}{\partial x}, \quad \dots \tag{68}$$

Since

$$m \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}}{(1 - v^2/c^2)^{\frac{3}{2}}} = \frac{d}{d\tau} \frac{mc^2}{(1 - v^2/c^2)^{\frac{1}{2}}},$$

these equations (68) possess the usual energy-integral

$$\frac{mc^2}{(1 - v^2/c^2)^{\frac{1}{2}}} + \chi = \text{const.}, \tag{69}$$

just as Einstein's equations of motion do. But (68) are not identical in form with Einstein's equations of motion, which are of the form

$$\frac{d}{dT}\left\{ \frac{m\dot{x}}{(1 - v^2/c^2)^{\frac{1}{2}}} \right\} = -\frac{\partial \chi}{\partial x}, \tag{70}$$

where T is a time-variable which sometimes plays the part of t, sometimes of τ. According to the developments of the present chapter, (70) rests on a confusion of ideas, and (68) is to be preferred. For one-dimensional motion, (70) and (68) coincide, if T is identified with τ. For

$$\frac{d}{d\tau} \frac{m\dot{x}}{(1 - \dot{x}^2/c^2)^{\frac{1}{2}}} = \frac{m\ddot{x}}{(1 - \dot{x}^2/c^2)^{\frac{3}{2}}}.$$

Einstein's equations of motion (70) are often derived by arguments which relate only to one-dimensional motion, and I suggest that (68) is the proper form. In general we have

$$\frac{m\ddot{x}}{(1 - v^2/c^2)^{\frac{1}{2}}} - \frac{d}{d\tau}\left\{ \frac{m\dot{x}}{(1 - v^2/c^2)^{\frac{1}{2}}} \right\} = -m\frac{\dot{y}(\dot{x}\dot{y} - \ddot{x}y) + \dot{z}(\dot{x}\ddot{z} - \ddot{x}\dot{z})}{c^2(1 - v^2/c^2)^{\frac{3}{2}}},$$

† These local Cartesian coordinates x, y, z, are not to be confused with the Lorentz coordinates (x, y, z), which form the vector **P**.

or, in vectors,

$$\frac{m\dot{\mathbf{v}}}{(1-v^2/c^2)^{\frac{1}{2}}} - \frac{d}{d\tau}\left\{\frac{m\mathbf{v}}{(1-v^2/c^2)^{\frac{1}{2}}}\right\} = m\frac{(\mathbf{v}\wedge\dot{\mathbf{v}})\wedge\mathbf{v}/c^2}{(1-v^2/c^2)^{\frac{1}{2}}}.$$

The difference thus vanishes only when $\dot{\mathbf{v}}$ and $\mathbf{v}$ are in the same direction.

120. Justification of a factor 2. It will have been noticed how essential a part has been played, in deriving these equations, by the factor 2 which occurs in the rate-of-change of mass term which was necessary to express an external force $\mathbf{F}$ in terms of a potential χ, namely in relations (21), (21′), Chapter VI. This coefficient 2 there looked anomalous. But its presence was needful in reducing the quasi-Lagrangian t-equations (15) and (15′) of the present chapter to the corresponding τ-equations (46) or (48). And it has also been seen to be essential in deducing the energy-integral direct from the t-equations (15) and (15′).

121. Hidden occurrence of parameter t_0. The philosophical significance of the analysis of this chapter is considerable. Starting with an equation of motion of a free test-particle derived in t-measure, we were led to construct a dynamics of unfamiliar form which contains no constants whatever save the conventional constant c. Transformation to τ-measure necessarily involves the occurrence of a parameter t_0, equal to the present value of t. This parameter t_0 occurs in the metric of the public hyperbolic space corresponding to the private Euclidean spaces used in t-measure, and consequently occurs in all *exact* statements of the equations of motion of a particle. But when the local Euclidean space is used instead of the public hyperbolic space, the parameter t_0 disappears explicitly to a first approximation, even when unrestricted velocities are envisaged. The resulting equations of motion differ significantly from Einstein's 'Special Relativity' dynamical equations, but have the same energy-integral.

It is due to this hidden occurrence of a parameter t_0 that we cannot set about deducing directly the equations of motion of a particle in τ-measure. For if we begin with a scale of time in which the fundamental particles are relatively stationary, we are fully entitled to choose a flat Euclidean space for the scene of events, and then we have no acquaintance with a parameter t_0 which would enable us

to construct exact equations. Further, since relative rest and abso-
lute simultaneity persist amongst the fundamental particles, the only
transformation at our disposal is that of spatial change of origin, and
this is not powerful enough to enable us to deduce the forms of
equations of motion. On the other hand, taking the substratum as
in relative motion, we have the variable t at our disposal, which is
not the time measured from an arbitrary time-zero, but the time
measured from the natural time-zero. This allows us to construct
acceleration in a general and non-arbitrary fashion, and so arrive at
a constant-free dynamics.

In this dynamics, kinetic energy is an invariant, taking the same
value whatever fundamental particle is chosen to provide the frame
of reference. Energy therefore has a definite meaning, unlike the
situation in Einstein's mechanics, where, not being an invariant but
being the fourth component of a vector, it changes its value with
changes of frame of reference. The meaning of this in the τ-mechanics
is that energy and motion should be reckoned in a frame moving
with the local characteristic velocity $V_0 = P/t$; in its application to
the universe at large this means that the nucleus of the nearest extra-
galactic nebula should be taken as origin. In τ-dynamics these
relatively stationary extra-galactic nuclei can be taken as providing
an absolute simultaneity. This justifies the Newtonian concept of
an absolute uniformly-flowing time, provided we identify this time
with our variable τ.

Particles or observers in uniform relative motion, in τ-measure,
relative to the stationary nebular nuclei, are *not* kinematically
equivalent to the nuclei themselves, and hence it cannot be expected
that the transformation from a nebular nucleus to a particle in
uniform motion (not zero) relative to that nucleus, will be represented
by Lorentz formulae. The arguments from covariance under Lorentz
transformation, used in special relativity to establish the form of the
equations of motion, are not therefore compelling, and it is not to be
wondered at that we have arrived at slightly different accurate forms
of the equations of motion in general, of the form (08).

The most interesting examples of the application of the t-dynamics
are to photons and the structure of spiral nebulae. We discuss the
former of these in the next chapter. The second problem we post-
pone until we have discussed gravitation in t-measure.

VIII

THE DYNAMICS OF LIGHT

122. Photons. In order to apply the dynamics we have now constructed to *light*, we shall regard light as consisting of particles called photons, which we shall consider as the limit of a material particle whose mass m tends to zero whilst its speed $|\mathbf{V}|$ tends to c, the velocity of light. The energy Ω of a material particle being given by the formula

$$\Omega = mc^2 \xi^{\frac{1}{2}} = mc^2 \frac{Z}{X^{\frac{1}{2}}Y^{\frac{1}{2}}} = mc^2 \frac{t - \mathbf{P}.\mathbf{V}/c^2}{(t^2 - \mathbf{P}^2/c^2)^{\frac{1}{2}}(1 - \mathbf{V}^2/c^2)^{\frac{1}{2}}},$$

where $\mathbf{P}$, $\mathbf{V}$ are its position vector and velocity at epoch t in the reckoning of a fundamental observer O, the energy of a photon at the origin O, as calculated by the observer O, will be

$$E = \lim_{\substack{m \to 0 \\ |\mathbf{V}| \to c}} \Omega_{\mathbf{P}=0} = \lim_{\substack{m \to 0 \\ |\mathbf{V}| \to c}} \frac{mc^2}{(1 - \mathbf{V}^2/c^2)^{\frac{1}{2}}}. \tag{1}$$

123. Frequency. This limit will have different values for different 'kinds' of light. It is readily seen that a new parameter, of the dimensions of an inverse time, is needed to characterize different 'kinds' of light. When $|\mathbf{V}| \to c$, the argument $Z = t - \mathbf{P}.\mathbf{V}/c^2$ tends, for a particle moving in the direction (λ, μ, ν), to the expression $t - \sum \lambda x/c$. To form the argument of a function expressing the propagation of light, we must combine $t - \sum \lambda x/c$ with a parameter rendering it dimensionless. We can do this as above, by combining it with the invariant $X^{\frac{1}{2}} = (t^2 - \mathbf{P}^2/c^2)^{\frac{1}{2}}$, and so arrive at the energy E. Or we can introduce another parameter n, of such a character that $n(t - \sum \lambda x/c)$ is an invariant. There should then be a relation between E and n, which it is the purpose of this chapter to explore. The parameter n is known in physics as the *frequency* of the light.

124. The justification (from the present point of view) of this name can be seen from the following. The energy Ω of a free particle in the substratum is not only an *invariant* but also *constant* along the trajectory of the free particle. It is natural to give the limit of Ω a similar property, and so we shall expect that E, the energy of a photon, will be both invariant and constant along the trajectory of the photon. If we give a similar property to the argument

$n(t - \sum \lambda x/c)$, then the values of this expression to the observer near whom the photon is emitted and to the observer near whom it is absorbed will be equal. Suppose that to the observer O' at a distant fundamental particle P_0, the frequency of a particular photon at epoch t' (to O') is n'. Let the observer O at the origin consider the frequency to be n_2 when it is absorbed near him at epoch t_2 (to O). Then from what has been said above since $x', y', z' = 0$ and $x, y, z = 0$, we have

$$n_2 t_2 = n' t'. \tag{2}$$

Applying the same relation to a photon of the same frequency emitted at (local) time $t' + dt'$, and received by O at (local) time $t_2 + dt_2$, we have

$$n_2 dt_2 = n' dt'. \tag{3}$$

The physical interpretation of this relation is that a number of pulses emitted at the rate n' per second during the small interval dt' all arrive at the rate n_2 during the small interval dt_2. Relation (3) expresses conservation of number of pulses; the pulses emitted are all absorbed. Hence the parameter n is called the *frequency* of the photon.

125. Formula for the Doppler effect. Let us now relate the ratio n_2/n' between reception frequency and emission frequency to the velocity $|V_0|$ of the fundamental particle or nebula P_0 which emitted the photon. If t is O's reckoning of the epoch of emission at P_0, then by the usual clock-running formula,† $t' = t(1 - V_0^2/c^2)^{\frac{1}{2}}$. But since, in O's reckoning, the photon left P_0 at epoch t and arrived at O at epoch t_2, travelling with speed c, we must have

$$t_2 = t + |P_0|/c.$$

But, by the recession law,

$$|P_0| = t|V_0|.$$

Hence

$$t_2 = t(1 + |V_0|/c).$$

Hence

$$t_2 = \frac{t'}{(1 - V_0^2/c^2)^{\frac{1}{2}}} (1 + |V_0|/c) = t' \left(\frac{1 + |V_0|/c}{1 - |V_0|/c} \right)^{\frac{1}{2}}.$$

Hence

$$\frac{dt_2}{dt'} = \left(\frac{1 + |V_0|/c}{1 - |V_0|/c} \right)^{\frac{1}{2}}.$$

Hence, by (3),

$$\frac{n_2}{n'} = \left(\frac{1 - |V_0|/c}{1 + |V_0|/c} \right)^{\frac{1}{2}}. \tag{4}$$

† In this chapter, where no ambiguities result, V_0, V_+, etc., will be used for $|V_0|$, $|V_+|$, etc.

This is the standard formula for the Doppler effect. We see that, due to the recession of P_0 with speed $|V_0|$, the frequency n_2 of the photon as it arrives at O is less than the frequency n' with which it left P_0, as calculated by O'.

If we identify diminishing frequency with increased *reddening* of the photon, we have the familiar red-shift due to recession.

For the sake of symmetry we make a similar calculation for a photon which leaves O with frequency n_1 (to O) and arrives at P_0 at epoch t' and with frequency n' (to O'). In this case, if t_1 is the epoch of departure of the photon from O, we have, t denoting O's reckoning of the epoch of arrival at O', as before,

$$t' = t(1 - V_0^2/c^2)^{\frac{1}{2}},$$

and
$$t = t_1 + |P_0|/c = t_1 + t|V_0|/c,$$

or
$$t(1 - |V_0|/c) = t_1.$$

Then
$$\frac{t'}{t_1} = \frac{(1 - V_0^2/c^2)^{\frac{1}{2}}}{1 - |V_0|/c} = \left(\frac{1 + |V_0|/c}{1 - |V_0|/c}\right)^{\frac{1}{2}},$$

whence
$$\frac{dt'}{dt_1} = \left(\frac{1 + |V_0|/c}{1 - |V_0|/c}\right)^{\frac{1}{2}},$$

and
$$\frac{n'}{n_1} = \frac{dt_1}{dt'} = \left(\frac{1 - |V_0|/c}{1 + |V_0|/c}\right)^{\frac{1}{2}},$$

Thus O' also sees a red-shift, since $n' < n_1$.

126. Relation between energy and frequency for a photon.

We now consider the relation between energy E and frequency n for a photon. The quantum theory asserts that E and n are proportional, that in fact $E = hn$, where h is a constant, called Planck's constant. Contemporary physics does not, however, distinguish between the two scales of time, so that it is not made clear whether n is to be regarded, in this formula, as measured on the t-scale, as is the n we have introduced in this chapter, or measured on the τ-scale. Disregarding this distinction for the time being, we see that a formula $E = hn$, with h constant, would imply that the arriving photon had less energy than when it started. For n_2 is less than n', and the red-shift should be accompanied by a loss of energy. This result, however, would be incompatible with the view that a photon may be regarded as a free particle. For, in the dynamics we have constructed, a free particle conserves its energy during its motion. Since, in addition, energy on our dynamics is an invariant, the same

for all fundamental observers, the energy of the photon at P_0, as calculated by O', will be equal to that calculated by O at the same event, and equal in turn to that calculated by O at the moment of reception. Thus, in spite of the red-shift, the energy of a photon should remain constant during its motion. The formula $E = hn$ would then need some revision. We shall show that the revision consists in taking h here to vary secularly with the epoch.

This amounts to a fundamental change in our view of the photon, as compared with that of contemporary physics. Its discrepancy with contemporary physics arises from the circumstance that in contemporary physics *energy* is not an invariant, but is the fourth component of a 4-vector, the momentum-energy vector. A consequence of this is that the measure of energy changes with the observer. Energy is no longer conserved in the strict sense, but is lost in the process of the emission of a photon by a distant nebula and its reception by ourselves. In our dynamics, on the other hand, energy is strictly conserved, not only in the experience of any one observer, but for all fundamental observers; all fundamental observers attach the same numerical value to any given store of energy.

The new view means that the energy of a photon depends not on its frequency alone but on its past history. It means that an atom is incapable of absorbing the photon whose energy is the energy-difference between two given stationary states of the atom unless the frequency is also correct.

127. Scrutiny of details. Before investigating further the consequences of this result, it is desirable to show that the limiting process by which we derive a photon from a material particle affords no loophole of escape.

Let us investigate in greater detail the career of a photon between a distant nebula and ourselves. For the sake of symmetry, we shall as before consider a photon which leaves our own galaxy, moves outwards and reaches a distant galaxy, where it is reflected, or absorbed and re-emitted, finally returning to our own galaxy.

Let the photon be emitted from ourselves at epoch t_1 with frequency n_1 and energy E_1; let it be reflected (or absorbed and re-emitted) at a distant galaxy P_0 at local time t'; and let it return to ourselves at our epoch t_2, with frequency (to us) n_2. Let n' be the frequency, in the frame of reference defined by the distant galaxy,

with which the photon is there received. Then since an observer on the distant galaxy can regard himself as at rest, n' will also be the frequency, in that frame, of the reflected or re-emitted photon. We proceed to calculate the actual energy of the photon at the different stages in its career.

Let V_1 be the velocity of the photon as it departs from O, with energy E_1, at time t_1. Then

$$E_1 = \lim_{\substack{|V_1| \to c \\ m \to 0}} \left\{ \frac{mc^2(t_1 - P \cdot V_1/c^2)}{(t_1^2 - P^2/c^2)^{\frac{1}{2}}(1 - V_1^2/c^2)^{\frac{1}{2}}} \right\}_{P=0} = \lim_{\substack{|V_1| \to c \\ m \to 0}} \frac{mc^2}{(1 - V_1^2/c^2)^{\frac{1}{2}}}.$$

When the photon reaches P_0, at time t to O, but just *before* reflection or absorption, let its energy be E_+, its speed V_+, as reckoned by O, and E'_+, V'_+ as reckoned by O'. Then

$$E_+ = \lim_{\substack{|V_+| \to c \\ m \to 0}} \frac{mc^2}{(1 - V_+^2/c^2)^{\frac{1}{2}}} \frac{t - P_0 \cdot V_+/c^2}{(t^2 - P_0^2/c^2)^{\frac{1}{2}}}.$$

Here
$$V_+ \to cP_0/|P_0|.$$

Hence
$$E_+ = \lim \frac{mc^2}{(1 - V_+^2/c^2)^{\frac{1}{2}}} \frac{t - |P_0|/c}{(t^2 - P_0^2/c^2)^{\frac{1}{2}}}$$
$$= \lim \frac{mc^2}{(1 - V_+^2/c^2)^{\frac{1}{2}}} \left(\frac{t - |P_0|/c}{t + |P_0|/c} \right)^{\frac{1}{2}}$$
$$= \lim \frac{mc^2}{(1 - V_+^2/c^2)^{\frac{1}{2}}} \left(\frac{1 - |V_0|/c}{1 + |V_0|/c} \right)^{\frac{1}{2}},$$

where V_0 is the recession-velocity of P_0. We have used $|V_0| = |P_0|/t$. Since it is of the essential nature of a particle that its mass m remains unchanged, we have from the above

$$\frac{E_+}{E_1} = \lim \left(\frac{1 - V_1^2/c^2}{1 - V_+^2/c^2} \right)^{\frac{1}{2}} \left(\frac{1 - |V_0|/c}{1 + |V_0|/c} \right)^{\frac{1}{2}}. \tag{5}$$

128. Integration of equation of motion. Though both V_1 and V_+ are equal to c in the limit, we are not entitled to assume $V_1 = V_+$. Instead, we must actually calculate the limit of the ratio

$$[(1 - V_1^2/c^2)/(1 - V_+^2/c^2)]^{\frac{1}{2}}$$

from the equation of motion or trajectory of the particle. This means integrating the equation of motion of a free particle in the limiting case when the speed tends to c. Of course, if we *assumed* that the above limit was unity, then (5) would give the usual Doppler effect formula, and the energies E_+ and E_1 could be considered as proportional to the corresponding frequencies. We should then have the

ordinary quantum theory of the energy of a photon. This was in fact the inference originally made by Whitrow,[†] who first made calculations of the present kind. He did not, however, treat the whole trajectory of the photon as that of a free particle, as we are doing.

Now the equation of motion of a free particle,

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(\mathbf{P} - \mathbf{V}\frac{Z}{Y}\right),$$

implies the corresponding scalar equation

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{2}}}\right) = -\frac{1}{X}\left(ct - c\frac{Z}{Y}\right).$$

It will be sufficient for our present purpose to use the latter equation. Multiplying both sides by $Y^{\frac{1}{2}}$ and taking $|\mathbf{V}| \sim c$, we get the approximate equation

$$\frac{d}{dt}\left(\frac{1}{Y^{\frac{1}{2}}}\right) \sim \frac{Z}{XY^{\frac{1}{2}}} = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}\log X^{\frac{1}{2}}.$$

The integral of this is $\dfrac{1}{Y^{\frac{1}{2}}} \sim \text{const.}\, X^{\frac{1}{2}}.$

This means that the ratio of the two values of $Y^{\frac{1}{2}}$ at two points on the trajectory of a photon is in the limit equal to the inverse ratio of the corresponding values of $X^{\frac{1}{2}}$. Applying this result to the trajectory of our photon from O to P_0, we get

$$\lim\left(\frac{1-V_1^2/c^2}{1-V_+^2/c^2}\right)^{\frac{1}{2}} = \frac{X_+^{\frac{1}{2}}}{X_1^{\frac{1}{2}}} = \frac{(t^2 - P_0^2/c^2)^{\frac{1}{2}}}{t_1} = \frac{t}{t_1}(1 - V_0^2/c^2)^{\frac{1}{2}}.$$

But we have seen that
$$t/t_1 = (1 - V_0/c)^{-1}.$$

Hence $$\lim\left(\frac{1-V_1^2/c^2}{1-V_+^2/c^2}\right)^{\frac{1}{2}} = \left(\frac{1+V_0/c}{1-V_0/c}\right)^{\frac{1}{2}}.$$

Inserting this in (5) we get

$$\frac{E_+}{E_1} = 1. \tag{6}$$

The photon thus does actually conserve its energy, in the reckoning of O, along its trajectory from O to P_0.

129. The distant observer. Next let us see how the observer O' at P_0 calculates the energy of the photon. He attributes to it

† G. J. Whitrow, *Quart. Journ. Math. (Oxford)*, **7**, 271, 1930.

a velocity, as it approaches P_0, say V'_+, where by Einstein's relative velocity formula

$$V'_+ = \frac{V_+ - V_0}{1 - V_+ V_0/c^2}.$$

From this, $\qquad 1 - V'^2_+/c^2 = \frac{(1 - V^2_+/c^2)(1 - V^2_0/c^2)}{(1 - V_+ V_0/c^2)^2}.$ \hfill (7)

Now in O''s reckoning, the energy with which the photon arrives at P_0, say E'_+, is given by

$$E'_+ = \lim_{\substack{m \to 0 \\ V'_+ \to c}} \frac{mc^2}{(1 - V'^2_+/c^2)^{\frac{1}{2}}} \left(\frac{t' - \mathbf{P'} \cdot \mathbf{V'_+}/c^2}{(t'^2 - \mathbf{P'}^2/c^2)^{\frac{1}{2}}} \right)_{P' = 0} = \lim \frac{mc^2}{(1 - V'^2_+/c^2)^{\frac{1}{2}}},$$

whence $\qquad \dfrac{E_+}{E'_+} = \lim_{V_+, V'_+ \to c} \left(\dfrac{1 - V'^2_+/c^2}{1 - V^2_+/c^2} \right)^{\frac{1}{2}} \left(\dfrac{1 - |\mathbf{V}_0|/c}{1 + |\mathbf{V}_0|/c} \right)^{\frac{1}{2}},$

or, using (7),

$$\frac{E_+}{E'_+} = \lim_{V_+ \to c} \frac{(1 - V^2_0/c^2)^{\frac{1}{2}}}{1 - V_0 V_+/c^2} \left(\frac{1 - V_0/c}{1 + V_0/c} \right)^{\frac{1}{2}} = 1. \hfill (8)$$

Thus $E'_+ = E_+$, and O' and O attribute the same energy to the photon as it arrives at P_0. This is fundamentally a consequence of the circumstance that Ω is an invariant, the same for all fundamental observers, and so the same for O and O', but it was necessary to show that in the passage from Ω to E this invariance was preserved.

130. In the frame in which P_0 is at rest, i.e. to O', the photon is reflected or re-emitted without change of frequency or energy. We therefore attribute the same value of m to the reflected, as to the incident photon. Thus

$$E'_+ = E'_-, \hfill (9)$$

where E'_- is the energy of the photon as it starts on its return journey from P_0 to O.

Now let E_- be the energy attributed by O to this photon as it leaves P_0, V_- its velocity. If V'_- is the velocity attributed by O', then as before

$$1 - V'^2_-/c^2 = \frac{(1 - V^2_-/c^2)(1 - V^2_0/c^2)}{(1 - V_- V_0/c^2)^2}. \hfill (10)$$

But $\qquad E'_- = \lim_{V'_- \to c} \dfrac{mc^2}{(1 - V'^2_-/c^2)^{\frac{1}{2}}}.$

Also $\qquad E_- = \lim \dfrac{mc^2}{(1 - V^2_-/c^2)^{\frac{1}{2}}} \dfrac{t - \mathbf{P}_0 \cdot \mathbf{V}_-/c^2}{(t^2 - \mathbf{P}^2_0/c^2)^{\frac{1}{2}}},$

in which we have to put

$$\mathbf{V}_- = -c\frac{\mathbf{P}_0}{|\mathbf{P}_0|}.$$

Hence

$$E_- = \lim_{\substack{m\to 0 \\ V_-\to -c}} \frac{mc^2}{(1-V_-^2/c^2)^{\frac{1}{2}}}\frac{t+|\mathbf{P}_0|/c}{(t^2-\mathbf{P}_0^2/c^2)^{\frac{1}{2}}}.$$

$$= \lim_{\substack{m\to 0 \\ V_-\to -c}} \frac{mc^2}{(1-V_-^2/c^2)^{\frac{1}{2}}}\left(\frac{1+V_0/c}{1-V_0/c}\right)^{\frac{1}{2}}.$$

Thus

$$\frac{E_-}{E_-'} = \lim\left(\frac{1-V_-'^2/c^2}{1-V_-^2/c^2}\right)^{\frac{1}{2}}\left(\frac{1+V_0/c}{1-V_0/c}\right)^{\frac{1}{2}}.$$

But, by (10),

$$\left(\frac{1-V_-'^2/c^2}{1-V_-^2/c^2}\right)^{\frac{1}{2}} = \lim_{V_-\to -c}\frac{(1-V_0^2/c^2)^{\frac{1}{2}}}{1-V_- V_0/c^2} = \left(\frac{1-V_0/c}{1+V_0/c}\right)^{\frac{1}{2}}.$$

Hence

$$\frac{E_-}{E_-'} = 1. \tag{11}$$

Thus, just as for the forward journey, O and O' attribute the same energy to the photon as it leaves P_0 on its return journey.

131. Return of the photon.

Lastly, for the arrival of the photon at O, with energy E_2 and speed V_2 (as calculated by O) we have

$$E_2 = \lim_{\substack{m\to 0 \\ V_2\to -c}} \frac{mc^2}{(1-V_2^2/c^2)^{\frac{1}{2}}}.$$

Applying the trajectory relation again, we have

$$\lim_{V_2, V_-\to -c}\left(\frac{1-V_2^2/c^2}{1-V_-^2/c^2}\right)^{\frac{1}{2}} = \frac{Xt}{Xt_2} = \frac{(t^2-\mathbf{P}_0^2/c^2)^{\frac{1}{2}}}{t_2} = \frac{t}{t_2}(1-V_0^2/c^2)^{\frac{1}{2}},$$

whilst

$$t_2 = t(1+V_0/c).$$

Hence

$$\lim\left(\frac{1-V_2^2/c^2}{1-V_-^2/c^2}\right)^{\frac{1}{2}} = \frac{(1-V_0^2/c^2)^{\frac{1}{2}}}{1+V_0/c} = \left(\frac{1-V_0/c}{1+V_0/c}\right)^{\frac{1}{2}},$$

and so

$$\frac{E_-}{E_2} = \lim\left(\frac{1-V_2^2/c^2}{1-V_-^2/c^2}\right)^{\frac{1}{2}}\left(\frac{1+V_0/c}{1-V_0/c}\right)^{\frac{1}{2}} = 1. \tag{12}$$

Thus, finally, the energy of the photon is conserved during its passage from P_0 to O.

132. Secular variation of Planck's constant.

We have now proved in detail, that if a photon can be considered as a free particle in the substratum, its energy at the different stages in its career satisfies the equalities

$$E_1 = E_+ = E_+' = E_-' = E_- = E_2. \tag{13}$$

Its frequency, however, satisfies the relations

$$\frac{n'}{n_1} = \frac{t_1}{t'} = \left(\frac{1-V_0/c}{1+V_0/c}\right)^{\frac{1}{2}}, \qquad \frac{n_2}{n'} = \frac{t'}{t_2} = \left(\frac{1-V_0/c}{1+V_0/c}\right)^{\frac{1}{2}}.$$

Hence
$$n_1 t_1 = n't' = n_2 t_2, \qquad (14)$$

and the frequency is less for O' than for O when emitting the photon, and less for O when receiving the photon again than for O'. If we telescope relations (13) into

$$E_1 = E' = E_2, \qquad (15)$$

and compare with (14), we see that to preserve a formula of the type

$$E = hn, \qquad (16)$$

we must have
$$E = h_1 n_1 = h' n' = h_2 n_2, \qquad (17)$$

where
$$\frac{h_1}{t_1} = \frac{h'}{t'} = \frac{h_2}{t_2}. \qquad (18)$$

This means that in reckoning the energy of a photon Planck's 'constant' must be taken as proportional to the epoch at the fundamental particle emitting or absorbing the photon. This statement refers to t-time.

Thus the photon in the course of emission at P_0 and reception at O retains a constant energy in spite of the reduction in frequency, as inferred on more general grounds in § 126. At P_0, on emission, it has the characteristic frequency (to O') of the atom which emits it; at O, it has no longer the characteristic frequency of an identical atom at O, although it has the same energy as when it started. Thus it cannot be absorbed by the identically similar atom at O, as it has no longer the right frequency. We deduce that the energy-frequency relation for a photon cannot be the same as the energy-frequency relation for an atomic transition.

133. Frequency of atomic transitions. We shall see later that the energy of an atom, W, though a time-invariant, depends on the value of the electrostatic charges of the electrons in it, which in turn depend on a calibration constant t_0; this is the measure of the epoch for which values of charges, etc., are standardized. Hence the constant of proportionality between an atomic energy-difference and the frequency of the corresponding photon emitted may depend on t_0. The frequency for the atomic transition must be supposed to be constant on the t-scale. We write then for an atomic transition

$$\Delta W = h_0 n_0. \qquad (19)$$

If this is the transition responsible for originating the photon of frequency n', emitted at P_0, then $n_0 = n'$ and $\Delta W = E' = h'n'$. Hence $h_0 = h'$. On reaching O the energy of the photon is still $E' = E_2 = h_2 n_2$, and thus

$$n_2 = \frac{h_0}{h_2} n_0 = \frac{h'}{h_2} n_0 = \frac{t'}{t_2} n_0. \tag{20}$$

Since the identically similar atom at O continues to absorb just the frequency n_0, $\neq n_2$, we see that this atom at O is incapable of absorbing the photon emitted at P_0, for it has not got the right frequency.

134. Expression in τ-measure. Consider the form which these relations take in τ-measure. If a small interval $\Delta\tau$ of τ-time corresponds to an interval Δt of t-time, then

$$\frac{\Delta t}{t} = \frac{\Delta\tau}{t_0}. \tag{21}$$

But if a photon of frequency n in t-measure has frequency ν in τ-measure, we have by counting light-pulses

$$n\Delta t = \nu\Delta\tau.$$

Hence $\quad\quad\quad\quad n t = \nu t_0.$

Hence if ν_1, ν', ν_2 correspond to n_1, n', n_2 respectively, we must have

$$\nu_1 = \frac{n_1 t_1}{t_0}, \quad\quad \nu' = \frac{n't'}{t_0}, \quad\quad \nu_2 = \frac{n_2 t_2}{t_0},$$

which, by (14), gives $\quad\quad \nu_1 = \nu' = \nu_2. \tag{22}$

The frequency in τ-measure is thus unaltered, which is consistent with the circumstance that in τ-measure the fundamental particles are relatively stationary, so that there is no Doppler shift. The energy of the photon is now given by

$$E_1 = h_1 n_1 = h_1 \frac{\nu_1 t_0}{t_1} = h_0 \nu_1, \tag{23}$$

and similarly $\quad\quad E' = h_0 \nu', \quad\quad E_2 = h_0 \nu_2. \tag{23'}$

In τ-measure, conservation of frequency goes with conservation of energy. The atomic transition relation becomes, on the other hand,

$$\Delta W = h_0 n_0 = h_0 \nu_0 \frac{t_0}{t}, \tag{24}$$

and thus, since ΔW must be constant, ν_0, the frequency in τ-measure of the radiation emitted or absorbed by the atom, is proportional

to t. This acceleration of atomic frequency results in an observable red-shift, just as in t-measure. But, now, the red-shift is caused by the acceleration in frequency of the atoms used to form a comparison spectrum. The energy-formula for a photon is now simply

$$E = h_0 \nu.$$

Thus the effect of going from t-time to τ-time is to transfer the secular variation from the photon to the atomic transition. As a coefficient for photon frequency in τ-measure, Planck's constant is actually constant. But the value at which it is stabilized depends on the normalization constant of the τ-scale concerned.

135. Evidence from angular momentum. Further theoretical evidence that on the t-scale Planck's constant h must be supposed to be proportional to t will emerge when we come to consider the angular momentum of an atom. In the t-dynamics, angular momentum is proportional to the absolute epoch t, but constant on the τ-scale. Hence when quantizing an angular momentum on the t-scale, by equating it to a multiple of $h/2\pi$, we have to assume h to vary secularly with the time, $h = h_0 t/t_0$; the angular momentum on the τ-scale then comes out a constant, and equal to a multiple of $h_0/2\pi$.

136. Observational evidence. The question now arises as to whether there is any observational evidence for the constancy of energy of a photon. To examine this we consider the question of the relation between the *observed* luminosity of a nebula or galaxy and its luminosity in a frame in which it is at rest.

Let us idealize a nebula or galaxy to the extent of considering it as a monochromatic emitter, of a certain effective wave-length. Let L' be its absolute luminosity in a frame moving with it, L_2 its absolute luminosity to an observer in our own galaxy, corrected of course for distance. Suppose that the luminosity L' arises from the emission of N' photons each of energy E', per unit of time, and that the observed luminosity L_2 arises from the reception of N_2 photons each of energy E_2, per unit of time. Then the number of photons emitted between local time t' and $t'+dt'$ is $N'dt'$, and the number received between times t_2 and t_2+dt_2 is $N_2 dt_2$. Since all the photons emitted in a small solid angle are absorbed, we have

$$N_2 dt_2 = N'dt'.$$

Also $$L_2 = N_2 E_2, \qquad L' = N'E'.$$

Hence
$$\frac{L'}{L_2} = \frac{N'}{N_2}\frac{E'}{E_2}.$$

On the old, classical theory, the energy of a photon is strictly proportional to its frequency, and we have

$$\frac{E'}{E_2} = \frac{n'}{n_2},$$

where n' is the frequency in the frame at rest relative to the distant galaxy, n_2 is the frequency observed at our own galaxy. By counting pulses emitted, we have

$$n'\,dt' = n_2\,dt_2,$$

and so
$$\frac{E'}{E_2} = \frac{n'}{n_2} = \frac{dt_2}{dt'} = \frac{t_2}{t'}.$$

As usual
$$t' = t(1 - V_0^2/c^2)^{\frac{1}{2}},$$

and
$$t_2 = t + V_0 t/c,$$

so that
$$\frac{t_2}{t'} = \left(\frac{1 + V_0/c}{1 - V_0/c}\right)^{\frac{1}{2}}.$$

Likewise
$$\frac{N'}{N_2} = \frac{dt_2}{dt'} = \frac{t_2}{t'}.$$

On the old view, then,
$$\frac{L'}{L_2} = \left(\frac{t_2}{t'}\right)^2.$$

Hubble calls the factor N'/N_2 the 'number'-effect and E'/E_2 the 'energy'-effect. The former arises from the dilution of the stream of photons consequent on the recession, the latter arises from the reduced energy-value of the red-shifted photon—on the classical theory.

On the other hand, according to the theory developed in this chapter, in spite of the red-shift we have

$$E' = E_2,$$

and so
$$\frac{L'}{L_2} = \frac{N'}{N_2} = \frac{t_2}{t'}.$$

We see that, on the older theory, the correcting factor is the square of the factor required on the new theory. On the new theory there is still the correction for the 'number'-factor, but no correction for the 'energy'-factor.

137. Hubble's results. Now Hubble has examined his nebular counts for given magnitude limits from two points of view. First,

taking the recession as real, and accordingly correcting for both factors; secondly taking the galaxies as relatively stationary, and so correcting only for the 'energy'-factor. As the two factors are numerically of the same size, it will be seen that Hubble's counts, uncorrected for motion but corrected for 'energy'-effect, correspond exactly to the new theory developed in this chapter, where we necessarily correct for motion but make no correction for energy. The origin of the correcting factor is ascribed differently, on Hubble's calculations, from the origin ascribed on the present theory; the name of the factor has changed, but the numerical consequence is the same. On Hubble's view, in his words, 'an "energy"-effect may be expected, regardless of the interpretation of red-shifts'; on our view, no energy-effect is to be expected, but the red-shifts are interpreted as velocity-shifts.

Hubble[†] then finds that the observed results are much closer to the theoretical predictions for the case of no motion:

'If the red-shifts are velocity-shifts, it follows that the universe is closed, having a finite volume and finite contents.' But 'the curvature required to remove the discrepancies is very great, and hence the radius of curvature is very small. Actually it is comparable with the radius of the observable region as defined with existing telescopes. Thus in order to save the velocity-shifts, we would be forced to conclude that the universe itself is so small that we are now observing a large fraction of the whole. . . . A radius of the dimensions necessary to save the velocity-shifts represents a mean density higher than 10^{-24} gram cm.$^{-3}$ This value is many times greater than even the maximum estimates of the smoothed-out density of the material concentrated in nebulae. . . . If the estimates of density were completely reliable, a radius of curvature of the necessary dimensions would be ruled out by the evidence. . . . On the other hand, if the interpretation as velocity-shifts is abandoned, we find in the red-shifts a hitherto unrecognized principle, whose implications are unknown.'

Again, in his Rhodes lectures,[‡] he writes:

'The familiar interpretation of red-shifts as velocity-shifts very seriously restricts not only the time-scale, the age of the universe, but the spatial dimensions as well. On the other hand, the alternative possible interpretation, that red-shifts are not velocity-shifts, avoids both difficulties, and presents the observable region as an insignificant sample of a universe that extends indefinitely in space and in time. . . . The disturbing features are all introduced by the recession-factors, by the assumption that red-shifts are velocity-shifts. The departure from a linear law of red-shifts, the departure from uniform distribution, the curvature demanded to restore homogeneity, the excess

† *The Realm of the Nebulae* (1936), chap. 8.
‡ *The Observational Approach to Cosmology* (1937), chap. 3.

material demanded by the curvature, each of these is merely the recession-factor in another form. . . . On the other hand, if the recession-factor is dropped, if red-shifts are not primarily velocity-shifts, the picture is simple and plausible.'

The theory of the present chapter resolves these difficulties. For what Hubble needs, to justify the more plausible world-picture, is just one correcting factor for nebular luminosities, not two. He erroneously calls this one factor the 'energy'-factor, and wishes to drop the 'velocity'-factor; actually it is the 'energy'-factor which is not required and the remaining factor that *is* required can now be retained as a 'recession'-factor, and the interpretation of the red-shifts as velocity-shifts is preserved. All his considerations are so much evidence that only one correcting factor is required. This we interpret as the recession-factor; the energy of the photon being unaltered by the red-shift, no 'energy'-factor is required.

This not only shows that the observations are compatible with the expansion of the universe. It is evidence for the correctness of the view developed in the present book, that energy is an invariant, not the time-component of a 4-vector, and that conservation of energy rules in the universe at large, in spite of the degradation of wave-length.

138. Luminosities in τ-measure. Let us examine the same question now using τ-time. Let $\mathscr{L}_2$ be the luminosity in τ-measure as observed at our galaxy, $\mathscr{L}'$ the luminosity to the observer located on the distant nebula. Let $\mathscr{N}_2$ be the number of photons emitted per second, each of energy $\mathscr{E}_2$, to the observer on our own galaxy; $\mathscr{N}'$, $\mathscr{E}'$ the corresponding quantities to the observer at the distant nebula. Then on the old theory, the 'energy'-effect still persists, and we have

$$\frac{\mathscr{E}'}{\mathscr{E}_2} = \frac{\nu'}{\nu_2},$$

but there is no 'number'-effect, since in τ-measure the distant galaxy is at rest relative to us. Thus, since

$$\mathscr{L}_2 = \mathscr{N}_2 \mathscr{E}_2, \qquad \mathscr{L}' = \mathscr{N}' \mathscr{E}',$$

and $$\mathscr{N}_2 = \mathscr{N}',$$

we have on the old theory

$$\frac{\mathscr{L}'}{\mathscr{L}_2} = \frac{\nu'}{\nu_2}.$$

But on the theory developed in this chapter,

$$\mathscr{E}_2 = \mathscr{E}'$$

and so $$\mathscr{L}' = \mathscr{L}_2.$$

139. Reconciliation with t-measure. We have to reconcile these calculations with our calculations in t-measure. The relationships are

$$L_2\,dt_2 = \mathscr{L}_2\,d\tau_2 \quad \text{i.e.} \quad L_2 t_2 = \mathscr{L}_2 t_0,$$

$$L'\,dt' = \mathscr{L}'\,d\tau', \quad \text{i.e.} \quad L't' = \mathscr{L}'t_0,$$

with
$$n_2\,dt_2 = \nu_2\,d\tau_2, \quad \text{i.e.} \quad n_2 t_2 = \nu_2 t_0,$$

and
$$n'\,dt' = \nu'\,d\tau', \quad \text{i.e.} \quad n't' = \nu't_0.$$

We are interested in the relation between the luminosities of the distant nebula in its own frame, in the two scales of time, i.e. in the ratio $\mathscr{L}'/L'$. According to the last group of formulae,

$$\frac{\mathscr{L}'}{L'} = \frac{t'}{t_0}.$$

If we use the corrections *according to the old theory* already obtained separately for $\mathscr{L}'$ and L' we have

$$\frac{\mathscr{L}'}{L'} = \frac{\mathscr{L}_2(\nu'/\nu_2)}{L_2(n'/n_2)^2} = \frac{\mathscr{L}_2}{L_2}\frac{\nu'}{n'}\frac{n_2}{\nu_2}\frac{n_2}{n'} = \frac{t_2}{t_0}\cdot\frac{t'}{t_0}\cdot\frac{t_0}{t_2}\cdot\frac{t'}{t_2} = \frac{t'^2}{t_0 t_2},$$

which contradicts the result just found. On the other hand, if we use the corrections *according to the new theory*, we get

$$\frac{\mathscr{L}'}{L'} = \frac{\mathscr{L}_2}{L_2(n'/n_2)} = \frac{\mathscr{L}_2}{L_2}\cdot\frac{n_2}{n'} = \frac{t_2}{t_0}\cdot\frac{t'}{t_2} = \frac{t'}{t_0},$$

which is correct. Thus the corrections on the old theory are inconsistent with one another, the inconsistency arising from the circumstance that in τ-measure, when the distant nebula is at relative rest, the old theory introduces irrationally an energy-shift, for which it gives no explanation. The relations

$$n_2\,dt_2 = n'\,dt' = n_2\,dt_2 = \nu_2\,d\tau_2,$$

and
$$n'\,dt' = \nu'\,d\tau',$$

and, lastly,
$$d\tau' = d\tau_2$$

compel the conclusion $\nu' = \nu_2$, which is only compatible with observation when the comparison atom at ourselves is accelerating in frequency. The relation $\nu' = \nu_2$ gives $h_0\nu' = h_0\nu_2$, so that as required the energy of the photon is conserved equally in τ-measure as in t-measure.

The cosmological evidence is thus strongly in favour of our conclusion that the energy of a photon remains constant in transit through the universe, in spite of the red-shift.

PART III

GRAVITATION

IX

STATISTICAL SYSTEMS

140. Need to consider statistical systems. We now take up again the question of the equation of motion of a free test-particle. We saw in §§ 68, 74 that the equation of motion of a free test-particle in the form

$$\frac{d\mathbf{V}}{dt} = \frac{Y}{X}(\mathbf{P} - \mathbf{V}t)G(\xi),\tag{1}$$

or in its 4-vector form

$$\frac{1}{Y^i}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^i}\right) = \frac{1}{X}\left(\mathbf{P} - \mathbf{V}\frac{Z}{Y}\right)G(\xi),\tag{1'}$$

would hold good not only in the presence of the substratum, but also in the presence of those more general systems which we have called *statistical* systems. It is obvious therefore that for the hydrodynamical substratum we must make a choice amongst all the different possible acceleration-laws (corresponding to different forms for $G(\xi)$) that will hold good in the presence of different statistical systems.

141. Boltzmann equation. Consider a statistical distribution of free particles of the form

$$f(x, y, z, t; u, v, w)\, dxdydzdudvdw,\tag{2}$$

where this number measures the number of particles moving at the instant t in the neighbourhood $dxdydz$ of (x, y, z) with velocities u, v, w in the neighbourhood $dudvdw$. Let

$$\mathfrak{g}(\mathbf{P}, t, \mathbf{V})\tag{3}$$

be the acceleration of the particle at P at epoch t, when moving with velocity V relative to the observer O at the origin. The 3-vector function $\mathfrak{g}$ must be of the form given by (1).

We first consider the condition that the population distribution (2) shall be consistent with the acceleration (3). We shall write (2) as $f(\mathbf{P}, t, \mathbf{V})$. Then when t changes to $t + \Delta t$, P changes to $\mathbf{P} + \mathbf{V}\Delta t$ and V to $\mathbf{V} + \mathfrak{g}\Delta t$, for any individual member. Put

$$t_1 = t + \Delta t,$$
$$\mathbf{P}_1 = \mathbf{P} + \Delta\mathbf{P} = \mathbf{P} + \mathbf{V}\Delta t,$$
$$\mathbf{V}_1 = \mathbf{V} + \Delta\mathbf{V} = \mathbf{V} + \mathfrak{g}\Delta t,$$

and write do for $dx\,dy\,dz$, $d\omega$ for $du\,dv\,dw$, with similar meanings for do_1 and $d\omega_1$. Then, in O's reckoning, all the particles inside $do\,d\omega$ at epoch t will be found inside $do_1\,d\omega_1$ at epoch t_1. Hence

$$f(\mathbf{P}, t, \mathbf{V})\,do\,d\omega = f(\mathbf{P}_1, t_1, \mathbf{V}_1)\,do_1\,d\omega_1.$$

A simple calculation shows that

$$\frac{do_1\,d\omega_1}{do\,d\omega} = \frac{\partial(x_1, y_1, z_1, u_1, v_1, w_1)}{\partial(x, y, z, u, v, w)} = 1 + \Delta t\left(\frac{\partial g_1}{\partial u} + \frac{\partial g_2}{\partial v} + \frac{\partial g_3}{\partial w}\right),$$

to a sufficient order, where $\mathbf{g} = (g_1, g_2, g_3)$. Hence

$$f\left(\mathbf{P} + \mathbf{V}\Delta t, t + \Delta t, \mathbf{V} + \mathbf{g}\Delta t\right)\left(1 + \Delta t\frac{\partial}{\partial \mathbf{V}}\cdot\mathbf{g}\right) = f(\mathbf{P}, \mathbf{V}, t).$$

This gives
$$\frac{\partial f}{\partial t} + \mathbf{V}\cdot\frac{\partial f}{\partial \mathbf{P}} + \mathbf{g}\cdot\frac{\partial f}{\partial \mathbf{V}} + f\left(\frac{\partial}{\partial \mathbf{V}}\cdot\mathbf{g}\right) = 0. \tag{4}$$

This is a generalization of the famous Boltzmann equation in the dynamical theory of gases, for the case when the acceleration may depend on the velocity and collisions are disregarded.

142. Use of Boltzmann equation to derive accelerations.

In its context in the dynamical theory of gases Boltzmann's equation is used to determine the distribution function when the acceleration function (or external field of force) is given. Here we shall use it in a reverse role. We are going to specialize the distribution function f so that it represents a distribution function described in the same way from every fundamental particle ($\mathbf{P}_0 = \mathbf{V}_0 t$) taken as origin, and then use this knowledge about f in (4) to provide information about $\mathbf{g}$.

143. Distribution formula.

We want to ensure that the statistics of the system are described in the same way by O and O', another fundamental observer in uniform motion relative to O. Consider a particle at $\mathbf{P}$ or (x, y, z) moving with velocity $\mathbf{V}$ or (u, v, w) at time t as counted by O. Let O' be moving with velocity $(U, 0, 0)$ relative to O. Then O' reckons the particle as at $\mathbf{P}'$ or (x', y', z'), moving with velocity $\mathbf{V}'$ or (u', v', w') at epoch t', where

$$x' = \frac{x - Ut}{(1 - U^2/c^2)^{\frac{1}{2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - Ux/c^2}{(1 - U^2/c^2)^{\frac{1}{2}}},$$

$$u' = \frac{u - U}{1 - uU/c^2}, \qquad v' = \frac{v(1 - U^2/c^2)^{\frac{1}{2}}}{1 - uU/c^2}, \qquad w' = \frac{w(1 - U^2/c^2)^{\frac{1}{2}}}{1 - uU/c^2}.$$

At the *same* epoch t, let O consider a neighbouring particle at $\mathbf{P}+d\mathbf{P}$, moving with velocity $\mathbf{V}+d\mathbf{V}$. The number of such particles is just $f(\mathbf{P}, t, \mathbf{V})\, do\, d\omega$. This particle is counted by O' at a different time $t'+\Delta t'$, as in the position $\mathbf{P}'+\Delta \mathbf{P}'$, and as moving with the velocity $\mathbf{V}'+\Delta \mathbf{V}'$, where

$$\Delta x' = \frac{dx}{(1-U^2/c^2)^{\frac{1}{2}}}, \quad \Delta y' = dy, \quad \Delta z' = dz, \quad \Delta t' = -\frac{(U/c^2)\, dx}{(1-U^2/c^2)^{\frac{1}{2}}},$$

$$\Delta u' = \frac{du(1-U^2/c^2)}{(1-uU/c^2)^2}, \qquad \Delta v' = \frac{dv(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2} + \dots du,$$

$$\Delta w' = \frac{dw(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2} + \dots du.$$

Retracing backwards the path of the particle *to the epoch* t', we shall find it at $\mathbf{P}'+d\mathbf{P}'$, moving with velocity $\mathbf{V}'+d\mathbf{V}'$, where

$$dx' = \Delta x' - u'\Delta t', \qquad dy' = \Delta y' - v'\Delta t', \qquad dz' = \Delta z' - w'\Delta t',$$

$$du' = \Delta u' - g_1'\Delta t', \qquad dv' = \Delta v' - g_2'\Delta t', \qquad dw' = \Delta w' - g_3'\Delta t',$$

g_1', g_2', g_3' being O''s reckoning of the acceleration components. The results of this calculation, as far as we require them, are

$$dx' = \frac{dx}{(1-U^2/c^2)^{\frac{1}{2}}}\{1+u'(U/c^2)\} = \frac{dx(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2},$$

$$dy' = dy + \dots dx, \qquad dz' = dz + \dots dx,$$

$$du' = \frac{du(1-U^2/c^2)}{(1-uU/c^2)^2} + \dots dx, \qquad dv' = \frac{dv(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2} + \dots du + \dots dx,$$

$$dw' = \frac{dw(1-U^2/c^2)^{\frac{1}{2}}}{1-uU/c^2} + \dots du + \dots dx.$$

(The coefficients indicated by $\dots$ are not needed in the sequel.)

We can now consider $(dx, \dots, dw)$, $(dx', \dots, dw')$ as two sets of coordinates, the second set the transform of the first. Then all the particles counted by O inside $do\, d\omega$ at time t will be counted by O' as inside $do'\, d\omega'$ at time t', where

$$\frac{do'\, d\omega'}{do\, d\omega} = \frac{\partial(dx', dy', dz', du', dv', dw')}{\partial(dx, dy, dz, du, dv, dw)}.$$

The value of this determinant is easily seen to be equal to the product of the terms in its leading diagonal, whence

$$\frac{do'\, d\omega'}{do\, d\omega} = \frac{(1-U^2/c^2)^{\frac{1}{2}}}{(1-uU/c^2)^5}.$$

Hence if $f'(P', t', V')$ is the distribution function found by O', we must have

$$f' \, do' d\omega' = f \, do \, d\omega = \frac{f(P, t, V)(1 - uU/c^2)^5}{(1 - U^2/c^2)^4} do' d\omega'.$$

But we are proposing for consideration statistical systems which are described in the same way by every fundamental observer O'. Hence f' must be of the same form as f, and so f must satisfy

$$f(\mathbf{P}', t', \mathbf{V}') = f(\mathbf{P}, t, \mathbf{V}) \frac{(1 - uU/c^2)^5}{(1 - U^2/c^2)^4}.$$

We have similar functional equations satisfied by f when $(U, 0, 0)$ is replaced by $(0, U, 0)$ and $(0, 0, U)$.

To solve this functional equation, put

$$f(\mathbf{P}, t, \mathbf{V}) = Y^{-4}\phi(\mathbf{P}, t, \mathbf{V}),$$

where, as usual, $Y = 1 - \mathbf{V}^2/c^2.$

Then $f(\mathbf{P}', t', \mathbf{V}') = Y'^{-4}\phi(\mathbf{P}', t', \mathbf{V}'),$

where $Y' = 1 - \mathbf{V}'^2/c^2.$

But $\dfrac{Y'}{Y} = \dfrac{1 - U^2/c^2}{(1 - uU/c^2)^2}.$

Hence the functional equation gives

$$\phi(\mathbf{P}', t', \mathbf{V}') = \phi(\mathbf{P}, t, \mathbf{V}).$$

Hence ϕ must be invariant under the Lorentz group of transformations from one fundamental observer to another. But the only invariants of this group are X and Z^2/Y, where, as usual,

$$X = t^2 - \mathbf{P}^2/c^2, \qquad Z = t - \mathbf{P} \cdot \mathbf{V}/c^2.$$

Hence $\phi(\mathbf{P}, t, \mathbf{V})$ is of the form

$$\phi(\mathbf{P}, t, \mathbf{V}) = \phi(X, Z^2/Y),$$

and so the statistical distribution function f which we require is of the form

$$f(\mathbf{P}, t, \mathbf{V}) \, dx\,dy\,dz\,du\,dv\,dw = \frac{\phi(X, Z^2/Y)}{Y^4} \, dx\,dy\,dz\,du\,dv\,dw. \tag{5}$$

This must be a pure number. Hence we can write it as

$$\frac{\phi(X, Z^2/Y)}{c^5 X^4 Y^4} \, dx\,dy\,dz\,du\,dv\,dw, \tag{6}$$

where now ϕ is of zero physical dimensions. The statistical system we are building up is, however, a pure construct, and it is to contain accordingly no so-called 'physical constants'. In fact the system so far constructed knows nothing of physical constants—there are none

in its make-up except the conventional c. But X and Z^2/Y are of dimensions $(time)^2$. Hence since we are unacquainted at this stage of our construction with any physical constant of the dimensions of a time, the arguments X and Z^2/Y of ϕ must occur as a ratio, otherwise ϕ could not be a pure number. Hence the distribution function may be written

$$f(\mathbf{P}, t, \mathbf{V})\, dx\,dy\,dz\,du\,dv\,dw = \frac{\psi(\xi)}{c^6 X^{\frac{3}{2}} Y^{\frac{3}{2}}}\, dx\,dy\,dz\,du\,dv\,dw, \tag{7}$$

where, as usual, $\xi = Z^2/XY$.

144. Non-uniqueness of a statistical system. The statistical systems thus specified are arbitrary to the extent of an arbitrary function ψ of a single variable ξ. Different functions ψ correspond to statistical systems of different structures. They are therefore not unique like the hydrodynamical substratum, which contained no arbitrary function in its description.

We saw that the acceleration of any free particle in the presence of a statistical system, and therefore the accelerations of the particle-members of a statistical system itself, are given by

$$\mathfrak{g}(\mathbf{P}, t, \mathbf{V}) = \frac{Y}{X}(\mathbf{P} - \mathbf{V}t) G(\xi). \tag{8}$$

The acceleration in a statistical system thus involves again one unknown function, $G(\xi)$, of the same variable ξ. The Boltzmann equation (4) provides a single relation between the distribution function $\psi(\xi)$ and the acceleration function $G(\xi)$. This fulfils our intuitive expectation that when a statistical system is specified by a particular function $\psi(\xi)$, then the acceleration of a free particle in its presence should be determinate.

145. Solution of the Boltzmann equation. We now introduce (7) for f and (8) for $\mathfrak{g}$ into the generalized Boltzmann equation (4). It will be a test of the accuracy of our argumentation that of the seven variables x, y, z, t, u, v, w occurring in this equation, the only combination that should eventually emerge should be the combination ξ.

Dividing the Boltzmann equation (4) by f we get, on inserting (7) and (8)

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{P}} + \frac{Y}{X} G(\xi)(\mathbf{P} - \mathbf{V}t) \cdot \frac{\partial}{\partial \mathbf{V}}\right) \log \frac{\psi(\xi)}{c^6 X^{\frac{3}{2}} Y^{\frac{3}{2}}} +$$
$$+ \frac{\partial}{\partial \mathbf{V}} \cdot \left\{\frac{Y}{X}(\mathbf{P} - \mathbf{V}t) G(\xi)\right\} = 0.$$

Putting $\qquad D \equiv \dfrac{\partial}{\partial t} + \mathbf{V} \cdot \dfrac{\partial}{\partial \mathbf{P}} + \dfrac{Y}{X} G(\xi)(\mathbf{P} - \mathbf{V}t) \cdot \dfrac{\partial}{\partial \mathbf{V}},$

the equation becomes

$$D \log \frac{\psi(\xi)}{X^{\frac{3}{2}}Y^{\frac{3}{2}}} + (\mathbf{P} - \mathbf{V}t) \cdot \frac{\partial}{\partial \mathbf{V}} \left\{ \frac{Y}{X} G(\xi) \right\} - 3t \frac{Y}{X} G(\xi) = 0. \qquad (9)$$

It simplifies the detailed reduction of this equation to note the following results:

$$\frac{\partial X}{\partial t} = 2t, \qquad \frac{\partial X}{\partial \mathbf{P}} = -\frac{2\mathbf{P}}{c^2}, \qquad \frac{\partial X}{\partial \mathbf{V}} = 0,$$

$$\frac{\partial Y}{\partial t} = 0, \qquad \frac{\partial Y}{\partial \mathbf{P}} = 0, \qquad \frac{\partial Y}{\partial \mathbf{V}} = -\frac{2\mathbf{V}}{c^2},$$

$$\frac{\partial Z}{\partial t} = 1, \qquad \frac{\partial Z}{\partial \mathbf{P}} = -\frac{\mathbf{V}}{c^2}, \qquad \frac{\partial Z}{\partial \mathbf{V}} = -\frac{\mathbf{P}}{c^2}.$$

From these we calculate

$$\frac{1}{\xi}\frac{\partial \xi}{\partial t} = \frac{2(X - tZ)}{XZ}, \qquad \frac{1}{\xi}\frac{\partial \xi}{\partial \mathbf{P}} = \frac{2(Z\mathbf{P} - X\mathbf{V})}{c^2 XZ}, \qquad \frac{1}{\xi}\frac{\partial \xi}{\partial \mathbf{V}} = \frac{2(\mathbf{V}Z - \mathbf{P}Y)}{c^2 YZ},$$

$$DX = 2Z,$$

$$DY = -2\frac{Y}{X}(tY - Z)G(\xi),$$

$$D\xi = -2\frac{Z}{X}(\xi - 1)\{1 + G(\xi)\},$$

$$(\mathbf{P} - \mathbf{V}t) \cdot \frac{\partial}{\partial \mathbf{V}} \left\{ \frac{Y}{X}G(\xi) \right\} = -\frac{2(tY - Z)}{X}G(\xi) + 2\frac{Z}{X}(1 - \xi)G'(\xi).$$

Insertion of these results in (9) yields

$$-\frac{3}{2}\frac{2Z}{X} - 2\frac{\psi'(\xi)}{\psi(\xi)}\frac{Z}{X}(\xi - 1)\{1 + G(\xi)\} + \frac{\frac{3}{2}}{Y}\frac{2Y}{X}(tY - Z)G(\xi) -$$

$$- 2\frac{G(\xi)}{X}(tY - Z) + 2\frac{Z}{X}(1 - \xi)G'(\xi) - 3t\frac{Y}{X}G(\xi) = 0.$$

The terms in t will be seen to cancel, and Z/X is a factor of the remaining terms. Removing this, we are left with

$$-2\frac{\psi'}{\psi}(1 + G)(\xi - 1) - 3(1 + G) - 2(\xi - 1)G' = 0,$$

or $\qquad\qquad \dfrac{\frac{3}{2}}{\xi - 1} + \dfrac{\psi'}{\psi} + \dfrac{G'}{1 + G} = 0.$

As expected, the only variable surviving is ξ. The relation we have found between G and ψ integrates in the form

$$(\xi-1)^{\mathfrak{i}}\psi(\xi)\{1+G(\xi)\} = \text{const.}$$

We shall call this constant $-C$ and write the result

$$G(\xi) \equiv -1 - \frac{C}{(\xi-1)^{\mathfrak{i}}\psi(\xi)}. \tag{10}$$

146. Formula (10) gives the connexion between the population-distribution in a statistical system, as measured by $\psi(\xi)$, and the acceleration of a free particle in the system, as measured by $G(\xi)$. Accordingly, the acceleration of a free particle in the presence of the statistical distribution (7) is given, by (1), by

$$\frac{d\mathbf{V}}{dt} = -\frac{Y}{X}(\mathbf{P}-\mathbf{V}t) - C\frac{(\mathbf{P}-\mathbf{V}t)}{(\xi-1)^{\mathfrak{i}}}\frac{Y}{X\psi(\xi)}. \tag{11}$$

147. Physical interpretation. Determination of $G(\xi)$ for a substratum. The question now arises: What are the kinematical and physical interpretations of this formula? What does it mean?

It represents the acceleration of a free particle as the sum of two components. The first,

$$-\frac{Y}{X}(\mathbf{P}-\mathbf{V}t), \tag{12}$$

is independent of $\psi(\xi)$ and so independent of the population of the statistical system. The second,

$$-C\frac{\mathbf{P}-\mathbf{V}t}{(\xi-1)^{\mathfrak{i}}}\frac{Y}{X\psi(\xi)}, \tag{13}$$

depends on $\psi(\xi)$, and depends moreover on a constant of integration C, which also requires interpretation. It should be remembered that the arguments of the present chapter have been so far purely kinematical, without any appeals to physical law.

Now the acceleration components (12) and (13) will hold good whatever superposition of a statistical system on a hydrodynamical substratum we care to consider. The hydrodynamical substratum is indeterminate to an arbitrary multiplier B in its description, and we can superpose a substratum of any B on a statistical system of any $\psi(\xi)$, and still obtain the acceleration components (12) and (13). The substratum and any statistical system may be considered as independent components of what we may call 'mixed' systems—

systems which are a superposition of hydrodynamical and statistical systems—and it therefore suggests itself that the acceleration component (12), which is independent of $\psi(\xi)$, is the component to be associated with the substratum above, and that the acceleration component (13), which depends on $\psi(\xi)$, is the component to be associated with a pure statistical system. This would mean that the acceleration in the presence of a substratum alone is given by (12), i.e. by $G(\xi) \equiv -1$, the form of $G(\xi)$ we have adopted in Chapter V.

148. Relation to gravitation. Let us see if there are any confirmatory arguments. In the first place, component (13) tends to zero as $\psi(\xi) \to \infty$. The meaning of $\psi(\xi) \to \infty$ is that the statistical system becomes more and more closely packed, and therefore in a sense more and more free from spaces between the particles, i.e. more and more of hydrodynamical character. This is in agreement with our suggested interpretation as far as it goes. But I do not find it of itself very convincing. A much more convincing argument is obtained by investigating the structure of the component (13).

We have

$$\xi - 1 = \frac{Z^2 - XY}{XY} = \frac{1}{XY}\left\{\frac{(\mathbf{P} - \mathbf{V}t)^2}{c^2} - \frac{(\mathbf{P} \wedge \mathbf{V})^2}{c^4}\right\}. \tag{14}$$

For $|\mathbf{V}| \ll c$, $|\mathbf{P}| \ll ct$, the component (13) reduces approximately to

$$-C \frac{\mathbf{P} - \mathbf{V}t}{|\mathbf{P} - \mathbf{V}t|^3}\left(\frac{c^3t}{\psi(1)}\right), \tag{15}$$

whilst the component (12) reduces approximately to

$$-\frac{\mathbf{P} - \mathbf{V}t}{t^2}. \tag{16}$$

Now the vector $\mathbf{P} - \mathbf{V}t$ is the vector joining to $\mathbf{P}$ the apparent centre $\mathbf{V}t$ of the system to an observer moving with the velocity $\mathbf{V}$ at P; for $\mathbf{V}t$ is the position of the fundamental particle relative to which the free particle under consideration is at rest. If we write $\mathbf{P} - \mathbf{V}t = \mathbf{r}$, the component (13) becomes an inverse square acceleration

$$-C \frac{c^3t}{\psi(1)} \frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{17}$$

and the component (12) becomes the linear acceleration

$$-\frac{\mathbf{r}}{t^2}. \tag{18}$$

Now we have previously, in Chapter V, compared this term with the Newtonian attraction due to the matter of the substratum between P and the apparent centre of spherical symmetry to the observer at P. We saw that since the density of the substratum near the centre was mB/c^3t^3, where m is the mass of a fundamental particle, therefore the mass of the sphere enclosed between P and the apparent centre is $(4\pi/3)r^3mB/c^3t^3$, the attraction at P due to this is

$$-\gamma\frac{4\pi}{3}r^3\frac{mB}{c^3t^3}\cdot\frac{1}{r^2},$$

where γ is the Newtonian 'constant' of gravitation, and that if we identify this attractive acceleration with component (12) we get

$$-\gamma\frac{4\pi}{3}r^3\frac{mB}{c^3t^3}\frac{1}{r^2}=-\frac{r}{t^2},$$

or
$$\gamma=\frac{c^3t}{(4\pi/3)mB}=\frac{c^3t}{M_0}. \tag{19}$$

It follows that the component (13) may be written

$$-\gamma\frac{C}{\psi(1)}M_0\frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{20}$$

which is the Newtonian attraction due to a point condensation of mass

$$\frac{CM_0}{\psi(1)} \tag{21}$$

at the apparent centre.

149. Emergence of the inverse square law of gravitation.

The component (13) thus expresses an inverse square attraction towards the apparent centre of the statistical system, for relatively near particles. No assumption equivalent to the introduction of an inverse square law was made in its derivation, and we have therefore obtained an approach to the law of gravitation by purely kinematic methods. Moreover, not only has the use of the Boltzmann equation compelled the emergence of an inverse square law, but it has thrown up a constant C, whose presence is equivalent, by (21) to the appearance of a parameter we may call the *gravitational mass* of the condensation at the apparent centre $P-Vt = 0$. The constant C measures the gravitational mass of the condensation implied at the centre $P-Vt = 0$ of the statistical system characterized by $\psi(\xi)$, in terms of $M_0/\psi(1)$ as a unit.†

† $M_0/\psi(1)$ may be interpreted physically as the mean apparent mass of the substratum per fundamental particle.

150. If we now rewrite, for greater clarity, relation (10) in the form

$$J(\xi) = -1 - \frac{C}{(\xi-1)^{\mathbf{1}}\psi(\xi)}, \tag{22}$$

we can say that the first term on the right here represents the part of the cosmical field of acceleration which is independent of $\psi(\xi)$, and so gives the value $G(\xi) \equiv -1$ corresponding to a substratum alone.

151. An argument of this type seems essential to deduce the acceleration of a free particle in the presence of the substratum alone. For we have to recognize that (1) or (1') is the equation of motion in the presence of *any* system satisfying the equivalence of every pair of fundamental observers, and therefore we have to devise a method of eliminating the possibility of the presence of statistical systems. We cannot introduce this negative consideration directly, but must trace the consequences of the system's including a statistical component, and then eliminate these consequences. We are repaid for our trouble by the unexpected emergence, of its own accord, of a form of the inverse square law of gravitation, and of the notion of gravitational mass.

152. Confirmatory argument. The foregoing argument can now be clinched by the consideration that when C reduces to zero, $J(\xi)$ reduces to -1; thus when the gravitational mass of the condensation is taken to be zero, the statistical component is absent, and the system reduces to a pure substratum, for which accordingly

$$G(\xi) \equiv -1.$$

In the next chapter we shall use the results of the present chapter to suggest a form of the law of gravitation applicable to any condensation.

THE INVERSE SQUARE LAW OF GRAVITATION

153. Construction of a potential function. We have seen that the statistical system behaves with respect to the particle at P at epoch t, moving with velocity V, as if it possessed a singularity at the place $P = Vt$ or $\xi = 1$. It is true, in fact, that the acceleration function $G(\xi)$ of equation (10), Chapter IX, which characterizes a statistical system, has a singularity wherever $\xi = 1$, and the free particle at P, t, V must be supposed to be principally under the influence of the nearest singularity. We shall put

$$m_2 = \frac{CM_0}{\psi(1)}, \tag{1}$$

and call m_2 the gravitational mass of the singularity.

The dynamical mass of the free particle is $m_1 \xi^{\frac{1}{2}}$. We therefore rewrite the equation of motion (1') of Chapter IX by multiplying it through by $m_1 \xi^{\frac{1}{2}}$, in the form

$$\frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}\left(m_1 \xi^{\frac{1}{2}} \frac{V}{Y^{\frac{1}{2}}}\right) = -\frac{m_1 \xi^{\frac{1}{2}}}{X}\left(P - V\frac{Z}{Y}\right) - \frac{m_1 m_2}{M_0} \frac{\psi(1)}{\psi(\xi)} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{3}{2}}}\left(P - V\frac{Z}{Y}\right)\frac{1}{X} +$$
$$+ \frac{V}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m_1 \xi^{\frac{1}{2}}). \tag{2}$$

Comparing this with equation (4) of Chapter VI, we see that the free particle in the presence of the statistical system is acted on by a force F, given by

$$F = -\frac{m_1 m_2}{M_0} \frac{\psi(1)}{\psi(\xi)} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{3}{2}}}\left(P - V\frac{Z}{Y}\right)\frac{1}{X} + \frac{V}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m_1 \xi^{\frac{1}{2}}). \tag{3}$$

The function $\psi(\xi)$ is at our disposal. We shall choose it so that the condensations in the statistical system round each singularity $\xi = 1$ correspond as nearly as possible to massive particles. From the results in the preceding chapter, this will be so if we take

$$\psi(\xi) = \text{const.} = \psi(1).$$

This is in effect a definition of what we mean by a gravitating particle. Then F reduces to

$$F = -\frac{m_1 m_2}{M_0} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{3}{2}}}\left(P - V\frac{Z}{Y}\right)\frac{1}{X} + \frac{V}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m_1 \xi^{\frac{1}{2}}). \tag{4}$$

We now seek to represent this force by a potential function χ. By equation (21), Chapter VI, this will be possible if a function χ can be found so that

$$\mathbf{F} = -\frac{\partial \chi}{\partial \mathbf{P}} + 2\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m_1 \xi^{\frac{1}{2}}). \qquad (5)$$

This requires

$$\frac{\partial \chi}{\partial \mathbf{P}} = \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left(\mathbf{P}-\mathbf{V}\frac{Z}{Y}\right)\frac{1}{X} + \frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m_1\xi^{\frac{1}{2}}), \qquad (6)$$

and there will also be the time-component (scalar) relation

$$-\frac{1}{c}\frac{\partial \chi}{\partial t} = \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left(ct-c\frac{Z}{Y}\right)\frac{1}{X} + \frac{c}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m_1\xi^{\frac{1}{2}}). \qquad (6')$$

We want $\chi/m_1 c^2$ to be a homogeneous function in $\mathbf{P}$ and ct of dimensions zero, and therefore we require

$$t\frac{\partial \chi}{\partial t} + \mathbf{P}\cdot\frac{\partial \chi}{\partial \mathbf{P}} = 0. \qquad (7)$$

This gives

$$\frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}c^2}{(\xi-1)^{\frac{1}{2}}}\left(-X+\frac{Z^2}{Y}\right)\frac{1}{X} - c^2\frac{Z}{Y}\frac{d}{dt}(m_1\xi^{\frac{1}{2}}) = 0,$$

i.e., since $Z^2/XY = \xi$,

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m_1\xi^{\frac{1}{2}}) = \frac{m_1 m_2}{M_0}\frac{1}{X^{\frac{1}{2}}(\xi-1)^{\frac{1}{2}}}. \qquad (8)$$

Introducing this into (6) we get

$$\frac{\partial \chi}{\partial \mathbf{P}} = \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left\{\frac{1}{X}\left(\mathbf{P}-\mathbf{V}\frac{Z}{Y}\right)+\frac{\mathbf{V}}{Z}\left(\frac{Z^2}{XY}-1\right)\right\}$$

$$= \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left(\frac{\mathbf{P}}{X}-\frac{\mathbf{V}}{Z}\right), \qquad (9)$$

and similarly, from (6'),

$$-\frac{\partial \chi}{c\,\partial t} = \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left\{\frac{1}{X}\left(ct-c\frac{Z}{Y}\right)+\frac{c}{Z}\left(\frac{Z^2}{XY}-1\right)\right\}$$

$$= \frac{m_1 m_2}{M_0}\frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}\left(\frac{ct}{X}-\frac{c}{Z}\right). \qquad (10)$$

But, by actual differentiation,

$$\frac{\partial \xi^{\frac{1}{2}}}{\partial \mathbf{P}} = \frac{\xi^{\frac{1}{2}}}{c^2}\left(-\frac{\mathbf{V}}{Z}+\frac{\mathbf{P}}{X}\right),$$

$$-\frac{\partial \xi^{\frac{1}{2}}}{c\,\partial t} = \frac{\xi^{\frac{1}{2}}}{c^2}\left(-\frac{c}{Z}+\frac{ct}{X}\right).$$

Hence

$$\frac{\partial \chi}{\partial \mathbf{P}} = \frac{m_1 m_2 c^2}{M_0} \frac{1}{(\xi-1)^{\frac{1}{2}}} \frac{\partial \xi^{\frac{1}{2}}}{\partial \mathbf{P}} = -\frac{m_1 m_2 c^2}{M_0} \frac{\partial}{\partial \mathbf{P}} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}, \qquad (11)$$

and

$$\frac{\partial \chi}{c\,\partial t} = \frac{m_1 m_2 c^2}{M_0} \frac{1}{(\xi-1)^{\frac{3}{2}}} \frac{\partial \xi^{\frac{1}{2}}}{c\,\partial t} = -\frac{m_1 m_2 c^2}{M_0} \frac{\partial}{c\,\partial t} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}. \qquad (12)$$

These are compatible, and give for χ the form

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{\xi^{\frac{1}{2}}}{(\xi-1)^{\frac{1}{2}}}. \qquad (13)$$

This, then, is the value of the potential χ suitable for describing the field of force holding in the presence of the statistical system.

154. Mutual potential of any two gravitating particles. We want to use the result just obtained to suggest the form of the potential due to a single gravitating particle. In (13), ξ is a function of $\mathbf{V}$, as well as of $\mathbf{P}$ and t, and the position of the singularity to which it refers is $\mathbf{V}t$, where the mass m_2 is supposed situated. Written out, (13) is

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{t-\mathbf{P}\cdot\mathbf{V}/c^2}{\{(t-\mathbf{P}\cdot\mathbf{V}/c^2)^2-(t^2-\mathbf{P}^2/c^2)(1-\mathbf{V}^2/c^2)\}^{\frac{1}{2}}},$$

and in this the epoch t refers to events at m_1. We shall therefore replace t by t_1, and $\mathbf{P}$ by $\mathbf{P}_1$, but $\mathbf{V}$ we shall replace by $\mathbf{P}_2/t_2$. We get

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{t_1 t_2 - \mathbf{P}_1\cdot\mathbf{P}_2/c^2}{\{(t_1 t_2 - \mathbf{P}_1\cdot\mathbf{P}_2/c^2)^2-(t_1^2-\mathbf{P}_1^2/c^2)(t_2^2-\mathbf{P}_2^2/c^2)\}^{\frac{1}{2}}}.$$

Writing

$$X_1 = t_1^2 - \mathbf{P}_1^2/c^2, \qquad X_2 = t_2^2 - \mathbf{P}_2^2/c^2, \qquad (14)$$

$$X_{12} = t_1 t_2 - \mathbf{P}_1\cdot\mathbf{P}_2/c^2, \qquad (14')$$

we have

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{X_{12}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}. \qquad (15)$$

We shall examine whether this can be taken to represent the potential energy of m_1 in the presence of m_2.

155. Properties of the gravitational potential χ. (1) We first note that χ is invariant in form and value whatever fundamental observer is chosen as origin. For X_1, X_2, X_{12} are 4-scalars.

(2) It will be found to satisfy the wave-equations

$$\frac{\partial^2 \chi}{\partial x_1^2} + \frac{\partial^2 \chi}{\partial y_1^2} + \frac{\partial^2 \chi}{\partial z_1^2} = \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t_1^2}, \qquad (16)$$

$$\frac{\partial^2 \chi}{\partial x_2^2} + \frac{\partial^2 \chi}{\partial y_2^2} + \frac{\partial^2 \chi}{\partial z_2^2} = \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t_2^2}. \qquad (16')$$

(3) It will be found to reduce to the Newtonian potential with 'constant' of gravitation $\gamma = c^3 t/M_0$ when either of the particles is taken as origin. For let the observer be at the mass m_2, so that $P_2 = 0$. Then χ reduces to

$$\chi = -m_1 m_2 \frac{c^3 t_1}{M_0} \frac{1}{|P_1|} = -\frac{\gamma_1 m_1 m_2}{|P_1|}. \tag{17}$$

This is independent of an assignment of a value to the epoch t_2 at m_2.

Again, if $|P_1-P_2|$ is small compared with either ct_1 or ct_2, and if, therefore, we can take the epoch t_1 at P_1 equal to the epoch t_2 at P_2, (P_1 and P_2 being close together), then since

$$(t_1 t_2 - P_1 . P_2/c^2)^2 - (t_1^2 - P_1^2/c^2)(t_2^2 - P_2^2/c^2) = \frac{(t_1 P_2 - t_2 P_1)^2}{c^2} - \frac{(P_1 \wedge P_2)^2}{c^4},$$

χ reduces to

$$\chi = -\frac{\gamma m_1 m_2}{|P_2 - P_1|}, \tag{18}$$

where $\gamma = c^3 t/M_0$ and t is the value of the common epoch at P_1 and P_2.

(4) When $|P_2| \to ct_2$, so that the attracting particle m_2 is near the boundary of the substratum, the potential χ tends to a constant, independent of the position of P_1. We have in fact

$$\chi_{|P_2| \to ct_2} \to -\frac{m_1 m_2 c^2}{M_0}. \tag{19}$$

The field at P_1 produced by m_2 at P_2 is therefore zero, approximately, and so the field of a particle on the confines of the substratum corresponds to the field of a Newtonian particle at infinity. We shall see this directly in a moment.

(5) By the general theory of Chapter VI, the 'force' on m_1 at P_1 produced by the potential χ is given by

$$F = -\frac{\partial \chi}{\partial P_1} + \frac{2}{Y_1^2} \frac{V}{Y_1^2} \frac{d}{dt_1}(m_1 \xi_1^2).$$

The part of this represented by the gradient of χ is given by

$$-\frac{\partial \chi}{\partial P_1} = +\frac{m_1 m_2 c^2}{M_0} \frac{\partial}{\partial P_1} \frac{X_{12}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}$$

$$= \frac{m_1 m_2}{M_0} \frac{X_2}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}}(P_2 X_1 - P_1 X_{12}). \tag{20}$$

We call this the *attraction*.

(6) If we take the origin at P_2, so that $P_2 = 0$, the attraction reduces to

$$-m_1 m_2 \frac{c^3 t_1}{M_0} \frac{P_1}{|P_1|^3}, \quad \text{or} \quad -\gamma_1 m_1 m_2 \frac{P_1}{|P_1|^3}, \tag{21}$$

which is an exact inverse square force directed towards P_2, the *attracting* particle. Moreover, its value depends only on O's assignment of the epoch t_1 at P_1, and so is independent of the imposition of any relation between the epochs t_2 and t_1 at P_2 and P_1.

(7) If we take the origin at the attracted particle, $P_1 = 0$, the attraction reduces to

$$m_1 m_2 \frac{c^3 t_2}{M_0} \frac{P_2}{|P_2|^3}\left(1 - \frac{P_2^2}{c^2 t_2^2}\right). \tag{22}$$

This again is an inverse square force directed towards P_2, multiplied by a cosmical factor which is nearly unity in ordinary experience ($|P_2| \ll ct_2$), but which vanishes when the attracting particle is on the confines of the universe.

(8) The attraction (20) vanishes whatever the origin, when $X_2 = 0$, i.e. when $|P_2| = ct_2$, i.e. when the attracting particle is on the confines of the universe.

(9) Whilst (20) gives the attraction on P_1, the attraction on P_2 is given by

$$-\frac{\partial \chi}{\partial P_2} = \frac{m_1 m_2}{M_0} \frac{X_1}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}}(P_1 X_2 - P_2 X_{12}). \tag{23}$$

Thus for an arbitrary origin, the two attractions are not exactly equal and opposite. But the attraction on P_2, reckoned by P_1, is equal and opposite to the attraction on P_1, reckoned by P_2, when the epochs of reckoning are equal. This then is the sense in which Newton's third law, the law of equality of action and reaction, must be held to be true in the context of gravitation.

156. Final identification of χ. The above properties of χ are sufficient to show that χ is the expression of the inverse square law of gravitation in Lorentz-invariant form. It is unaltered in form and value under transformation from any one fundamental observer to any other. It was, of course, the supposed impossibility of expressing the inverse square law of gravitation in Lorentz-invariant form which led to the so-called general theory of relativity. It will be seen that the difficulties vanish as soon as we take into account the phenomenon of the expanding universe, and recognize the existence, at each point

of accessible space, of an absolute standard of rest. These considerations imply, in turn, a 'constant' of gravitation varying secularly with the epoch, in t-measure.

It remains to show that the form we have chosen for χ represents in fact the potential energy of the two particles. To show this we have recourse to the general equations of motion of the two particles m_1 and m_2 in one another's presence and in the presence of the substratum.

157. Equations of motion in t-measure. The general equations of motion of the two particles, by the general theory of Chapter VI, are

$$\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}\left(m_1\,\xi_1^{\frac{1}{2}}\frac{V_1}{Y_1^{\frac{1}{2}}}\right) = -\frac{m_1\,\xi_1^{\frac{1}{2}}}{X_1}\left(P_1-V_1\frac{Z_1}{Y_1}\right)-\frac{\partial\chi}{\partial P_1}+2\frac{V_1}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1\,\xi_1^{\frac{1}{2}}), \quad (24)$$

$$\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}\left(m_1\,\xi_1^{\frac{1}{2}}\frac{c}{Y_1^{\frac{1}{2}}}\right) = -\frac{m_1\,\xi_1^{\frac{1}{2}}}{X_1}\left(ct_1-c\frac{Z_1}{Y_1}\right)+\frac{\partial\chi}{c\,\partial t_1}+2\frac{c}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1\,\xi_1^{\frac{1}{2}}), \quad (24')$$

$$\frac{1}{Y_2^{\frac{1}{2}}}\frac{d}{dt_2}\left(m_2\,\xi_2^{\frac{1}{2}}\frac{V_2}{Y_2^{\frac{1}{2}}}\right) = -\frac{m_2\,\xi_2^{\frac{1}{2}}}{X_2}\left(P_2-V_2\frac{Z_2}{Y_2}\right)-\frac{\partial\chi}{\partial P_2}+2\frac{V_2}{Y_2^{\frac{1}{2}}}\frac{1}{Y_2^{\frac{1}{2}}}\frac{d}{dt_2}(m_2\,\xi_2^{\frac{1}{2}}), \quad (25)$$

$$\frac{1}{Y_2^{\frac{1}{2}}}\frac{d}{dt_2}\left(m_2\,\xi_2^{\frac{1}{2}}\frac{c}{Y_2^{\frac{1}{2}}}\right) = -\frac{m_2\,\xi_2^{\frac{1}{2}}}{X_2}\left(ct_2-c\frac{Z_2}{Y_2}\right)+\frac{\partial\chi}{c\,\partial t_2}+2\frac{c}{Y_2^{\frac{1}{2}}}\frac{1}{Y_2^{\frac{1}{2}}}\frac{d}{dt_2}(m_2\,\xi_2^{\frac{1}{2}}). \quad (25')$$

These equations are invariant in form for any fundamental observer O of the substratum. But they only determine an actual motion when we impose a relation between the two independent variables t_1 and t_2. To preserve the Lorentz-invariance of the equations, the desired relation should itself be Lorentz-invariant. It was suggested by G. L. Camm† that the appropriate relation is

$$X_1 = X_2. \quad (26)$$

For this is Lorentz-invariant, is symmetrical between the particles, and reduces approximately to $t_1 = t_2$ whenever P_1 and P_2 are not cosmically far from the observer. More cogent still is the consideration due to Whitrow, that the meaning of $X_1 = X_2$ is seen when we use the τ-scale of time, when it reduces to

$$\tau_1 = \tau_2.$$

This is highly appropriate, considering that, in the τ-scale of time, there is a public time, or absolute simultaneity, τ being, for any given event, independent of the observer chosen to describe the event.

† *Nature*, **155**, 754, 1945.

158. Identical relation between the equations of motion. The pair of equations (24) and (24') are not independent, for multiplying (24) scalarly by P_1 and (24') by ct_1 and subtracting, we recover the identity (7), which is in fact satisfied identically by χ as given by (15), since χ/mc^2 is of zero dimensions in the variables P_1 and ct_1.

159. Energy-integral. Next, multiply (24) scalarly by V_1 and (24') by c and subtract. We get

$$\frac{\partial \chi}{\partial t_1} + V_1 \cdot \frac{\partial \chi}{\partial P_1} + \frac{d}{dt_1}(m_1 c^2 \xi_1^{\frac{1}{2}}) = 0. \tag{27}$$

Similarly

$$\frac{\partial \chi}{\partial t_2} + V_2 \cdot \frac{\partial \chi}{\partial P_2} + \frac{d}{dt_2}(m_2 c^2 \xi_2^{\frac{1}{2}}) = 0. \tag{27'}$$

Hence if t is any parameter,

$$\frac{d\chi}{dt} = \left(\frac{\partial \chi}{\partial t_1} + V_1 \cdot \frac{\partial \chi}{\partial P_1}\right)\frac{dt_1}{dt} + \left(\frac{\partial \chi}{\partial t_2} + V_2 \cdot \frac{\partial \chi}{\partial P_2}\right)\frac{dt_2}{dt}$$

$$= -\frac{d}{dt}(m_1 c^2 \xi_1^{\frac{1}{2}} + m_2 c^2 \xi_2^{\frac{1}{2}}),$$

or

$$\chi + m_1 c^2 \xi_1^{\frac{1}{2}} + m_2 c^2 \xi_2^{\frac{1}{2}} = \text{const.} \tag{28}$$

Since $m_1 c^2 \xi_1^{\frac{1}{2}}$ and $m_2 c^2 \xi_2^{\frac{1}{2}}$ are the kinetic energies (including rest energy), Ω_1 and Ω_2, of the two particles, the foregoing is the energy-integral

$$\chi + \Omega_1 + \Omega_2 = \text{const.} \tag{29}$$

Thus $\chi(P_1, P_2, t_1, t_2)$ actually represents the joint gravitational potential energy of the pair of particles in one another's presence. This relation is to be interpreted by relating t_1 and t_2 according to (26).

160. Three-dimensional form of energy-formula. To interpret the intermediate equations (27), (27') substitute for $\partial \chi/\partial t_1$ from the Eulerian relation

$$t_1 \frac{\partial \chi}{\partial t_1} + P_1 \cdot \frac{\partial \chi}{\partial P_1} = 0. \tag{30}$$

We get

$$\frac{d}{dt_1}(m_1 c^2 \xi_1^{\frac{1}{2}}) = \left(-\frac{\partial \chi}{\partial P_1}\right) \cdot \left(V_1 - \frac{P_1}{t_1}\right), \tag{31}$$

and thus the rate of increase of kinetic energy of the particle m_1 is equal to the work done by the attraction $(-\partial \chi/\partial P_1)$ in pushing the particle relative to its immediate cosmic environment at the relative velocity $V_1 - P_1/t_1$. Similarly for particle m_2.

161. Actual acceleration. To obtain the actual acceleration of m_1, multiply (24') by V_1/c and subtract it from (24). We get

$$\frac{dV_1}{dt_1} = -\frac{Y_1}{X_1}(P_1 - V_1 t_1) + \left(-\frac{\partial \chi}{\partial P_1} - \frac{V_1}{c^2}\frac{\partial \chi}{\partial t_1}\right)\frac{Y_1}{m_1 \xi_1^4} \tag{32}$$

$$= -\frac{Y_1}{X_1}(P_1 - V_1 t_1) + \frac{Y_1}{m_1 \xi_1^4}\left(U - \frac{V_1 P_1}{c^2 t_1}\right) \cdot \left(-\frac{\partial \chi}{\partial P_1}\right), \tag{33}$$

where U is the idem-tensor and $V_1 P_1$ is a dyad.

Since the first term on the right-hand side is the acceleration of P_1 due to the substratum alone, the second term represents the component of acceleration due to the attraction.

162. Identities satisfied by χ. The gravitational potential χ satisfies a number of identities, amongst which are the identities which lead to the principles of linear and angular momentum for the pair of particles. Amongst these identities the following may be verified:

since
$$\frac{P_2 X_1 - P_1 X_{12}}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}} = -\frac{\partial}{\partial P_2}\frac{1}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}},$$

it follows from (20) that

$$-\frac{\partial \chi}{\partial P_1} = X_2 \frac{\partial}{\partial P_2}\left(\frac{\chi}{X_{12}}\right). \tag{34}$$

Similarly
$$-\frac{\partial \chi}{\partial P_2} = X_1 \frac{\partial}{\partial P_1}\left(\frac{\chi}{X_{12}}\right). \tag{34'}$$

Hence we have the tensor identities

$$-\frac{1}{X_2}\frac{\partial^2 \chi}{\partial P_1 \partial P_1} = \frac{\partial^2}{\partial P_1 \partial P_2}\left(\frac{\chi}{X_{12}}\right) = -\frac{1}{X_1}\frac{\partial^2 \chi}{\partial P_2 \partial P_2}. \tag{35}$$

Contracting,

$$-X_1 \nabla_1^2 \chi = X_1 X_2 \frac{\partial^2}{\partial P_1 . \partial P_2}\left(\frac{\chi}{X_{12}}\right) = -X_2 \nabla_2^2 \chi. \tag{36}$$

Similarly

$$-X_1 \frac{\partial^2 \chi}{\partial t_1^2} = X_1 X_2 \frac{\partial^2}{\partial t_1 \partial t_2}\left(\frac{\chi}{X_{12}}\right) = -X_2 \frac{\partial^2 \chi}{\partial t_2^2}. \tag{36'}$$

It follows from (16) and (16') that

$$\frac{\partial^2}{\partial P_1 . \partial P_2}\left(\frac{\chi}{X_{12}}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2}\left(\frac{\chi}{X_{12}}\right) = 0, \tag{37}$$

Further, we have the scalar identity

$$\frac{1}{X_2}\left(t_2\frac{\partial\chi}{\partial t_1}+\mathbf{P}_2\cdot\frac{\partial\chi}{\partial\mathbf{P}_1}\right)=\frac{\chi}{X_{12}}=\frac{1}{X_1}\left(t_1\frac{\partial\chi}{\partial t_2}+\mathbf{P}_1\cdot\frac{\partial\chi}{\partial\mathbf{P}_2}\right),\tag{38}$$

the vector identities involving vector products

$$\mathbf{P}_1\wedge\left(-\frac{\partial\chi}{\partial\mathbf{P}_1}\right)+\mathbf{P}_2\wedge\left(-\frac{\partial\chi}{\partial\mathbf{P}_2}\right)=0,\tag{39}$$

$$\frac{\mathbf{P}_2}{X_2}\wedge\left(-\frac{\partial\chi}{\partial\mathbf{P}_1}\right)+\frac{\mathbf{P}_1}{X_1}\wedge\left(-\frac{\partial\chi}{\partial\mathbf{P}_2}\right)=0,\tag{40}$$

and the 'straight' vector identity

$$-t_1\frac{\partial\chi}{\partial\mathbf{P}_1}-\frac{\mathbf{P}_1}{c^2}\frac{\partial\chi}{\partial t_1}-t_2\frac{\partial\chi}{\partial\mathbf{P}_2}-\frac{\mathbf{P}_2}{c^2}\frac{\partial\chi}{\partial t_2}=0.\tag{41}$$

163. Physical application of identities. We shall now show that identity (41) corresponds to the integral of linear momentum of the pairs of equations (24) and (25); and that (39) corresponds to the integral of angular momentum. For each may be used separately to eliminate the gravitational potential χ between the equations concerned.

164. The equation of linear momentum. Multiply (24) by t_1, (24') by $-\mathbf{P}_1/c$, and perform similar operations on (25) and (25'); then add the results. We get, on using (41),

$$\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\left\{t_1\frac{d}{dt_1}\left(\frac{\mathbf{V}}{Y_1^{\frac{1}{2}}}\right)-\mathbf{P}_1\frac{d}{dt_1}\left(\frac{1}{Y_1^{\frac{1}{2}}}\right)\right\}+$$

$$+\frac{m_1\xi_1^{\frac{1}{2}}}{X_1}\frac{Z_1}{Y_1}(\mathbf{P}_1-\mathbf{V}_1 t_1)-\frac{\mathbf{V}_1 t_1-\mathbf{P}_1}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1\xi_1^{\frac{1}{2}})$$

$$+\text{similar expression with symbols suffixed }2=0.\tag{42}$$

Now
$$t_1\frac{d}{dt_1}\left(\frac{\mathbf{V}}{Y_1^{\frac{1}{2}}}\right)-\mathbf{P}_1\frac{d}{dt_1}\left(\frac{1}{Y_1^{\frac{1}{2}}}\right)=\frac{d}{dt_1}\left(\frac{\mathbf{V}_1 t_1-\mathbf{P}_1}{Y_1^{\frac{1}{2}}}\right),$$

and
$$\frac{Z_1}{X_1 Y_1}=\frac{1}{Y_1}\frac{d}{dt_1}(\log X_1^{\frac{1}{2}}).$$

Hence the terms suffixed 1 in (42) come to

$$\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\left\{\frac{d}{dt_1}\left(\frac{\mathbf{V}_1 t_1-\mathbf{P}_1}{Y_1^{\frac{1}{2}}}\right)-\frac{\mathbf{V}t_1-\mathbf{P}_1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(\log X_1^{\frac{1}{2}}+\log m_1\xi_1^{\frac{1}{2}})\right\}.$$

The integrating factor of these terms is

$$\exp\left\{-\int^{t_1}\frac{d}{dt_1}(\log m_1\xi_1^{\frac{1}{2}}X_1^{\frac{1}{2}})\,dt_1\right\},$$

or
$$X_1^{-\frac{1}{2}}\xi_1^{-\frac{1}{2}}.$$

We therefore get

$$\sum_{1,2}\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}X_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\frac{d}{dt_1}\left(\frac{V_1t_1-P_1}{Y_1^{\frac{1}{2}}X_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}}\right)=0, \qquad (43)$$

or
$$\sum_{1,2}\frac{m_1\xi_1}{Y_1^{\frac{1}{2}}}X_1^{\frac{1}{2}}\frac{d}{dt_1}\left(\frac{V_1t_1-P_1}{Z_1}\right)=0. \qquad (43')$$

We shall transform this rigorously into τ-measure in the next chapter, and show that it leads to a form of the equation of linear momentum. In the meantime we notice that when $|V|\ll c$, $|P|\ll ct$, it reduces approximately to

$$\sum_{1,2}m_1t_1\frac{d}{dt_1}\left(V_1-\frac{P_1}{t_1}\right)=0, \qquad (44)$$

or, in τ-measure,
$$\sum_{1,2}m_1\frac{d}{d\tau_1}(v_1)=0. \qquad (45)$$

This is to be understood with the condition

$$\tau_1=\tau_2.$$

It thus states the principle of linear momentum in τ-measure for any pair of gravitating particles.

165. The equation of angular momentum. If we multiply equation (24) vectorially by P_1, (25) vectorially by P_2, add and use identity (39), we again eliminate the gravitational potential χ. We get

$$\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}\left(\frac{P_1\wedge V_1}{Y_1^{\frac{1}{2}}}\right)-\frac{m_1\xi_1^{\frac{1}{2}}}{X_1}\frac{Z_1}{Y_1}(P_1\wedge V_1)-\frac{P_1\wedge V_1}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1\xi_1^{\frac{1}{2}}) \quad (46)$$

$$+\text{similar terms suffixed } 2 = 0.$$

This may be written, as before,

$$\sum_{1,2}\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\left\{\frac{d}{dt_1}\left(\frac{P_1\wedge V_1}{Y_1^{\frac{1}{2}}}\right)-\frac{P_1\wedge V_1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(\log X_1^{\frac{1}{2}}+\log m_1\xi_1^{\frac{1}{2}})\right\}=0.$$

The integrating factor of each term is again

$$X_1^{-\frac{1}{2}}\xi_1^{-\frac{1}{2}}.$$

Hence
$$\sum_{1,2} \frac{m_1 \xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}} X_1^{\frac{1}{2}} \xi_1^{\frac{1}{2}} \frac{d}{dt_1}\left(\frac{\mathbf{P}_1 \wedge \mathbf{V}_1}{\xi_1^{\frac{1}{2}} X_1^{\frac{1}{2}} Y_1^{\frac{1}{2}}}\right) = 0, \tag{47}$$

or
$$\sum \frac{m_1 \xi_1}{Y_1^{\frac{1}{2}}} X_1^{\frac{1}{2}} \frac{d}{dt_1}\left(\frac{\mathbf{P}_1 \wedge \mathbf{V}_1}{Z_1}\right) = 0. \tag{47'}$$

This again we shall transform rigorously into τ-measure in the next chapter. In the meantime we notice that for $|\mathbf{V}| \ll c$, $|\mathbf{P}| \ll ct$, it reduces approximately to

$$\sum_{1,2} m_1 t_1 \frac{d}{dt_1}\left(\frac{\mathbf{P}_1 \wedge \mathbf{V}_1}{t_1}\right) = 0, \tag{48}$$

or, in τ-measure,
$$\sum_{1,2} m_1 \frac{d}{d\tau_1}(\mathbf{\Pi}_1 \wedge \mathbf{v}_1) = 0. \tag{49}$$

This is to be understood with the convention

$$\tau_1 = \tau_2.$$

It thus states the principle of angular momentum in τ-measure for any pair of gravitating particles.

166. Extension to n particles. If n gravitating particles are present in the substratum, we write for the potential χ

$$\chi = \sum_{r \neq s} \chi_{rs}, \tag{50}$$

where
$$\chi_{rs} = -\frac{m_r m_s c^3}{M_0} \frac{X_{rs}}{(X_{rs}^2 - X_r X_s)^{\frac{1}{2}}}. \tag{51}$$

Equations of the type (24), (24') with χ given by (50) hold for each particle. They are to be understood in the sense that the various time variables t_1, t_2,... are to be correlated by

$$X_1 = X_2 = ... = X_n. \tag{52}$$

By the same procedure as led to the energy-integral (28) we find the general energy-integral

$$\chi + \sum_r m_r \xi_r^{\frac{1}{2}} = \text{const.} \tag{53}$$

The identities of the types (39), (41) are satisfied by χ_{rs} in the variables suffixed r and s, and can be used for combining the complete set of equations of motion of the n particles with elimination of the χ_{rs}'s, and we get the differential forms of the principles of linear and angular momentum for the n particles in the forms

$$\sum \frac{m_r \xi_r^{\frac{1}{2}}}{Y_r^{\frac{1}{2}}} X_r^{\frac{1}{2}} \xi_r^{\frac{1}{2}} \frac{d}{dt_r}\left(\frac{\mathbf{V}_r t_r - \mathbf{P}_r}{Z_r}\right) = 0, \tag{54}$$

$$\sum \frac{m_r \xi_r^{\frac{1}{2}}}{Y_r^{\frac{1}{2}}} X_r^{\frac{1}{2}} \xi_r^{\frac{1}{2}} \frac{d}{dt_r}\left(\frac{\mathbf{P}_r \wedge \mathbf{V}_r}{Z_r}\right) = 0. \tag{55}$$

167. Keplerian problem. An important special case is when one of a pair of particles is relatively very massive compared with the other $(m_2 \gg m_1)$. In that case we can consider the attracting particle m_2 as permanently located at a fundamental particle, so that $P_2 = 0$ permanently. Then

$$\chi = -\frac{m_1 m_2 c^3 t}{M_0}\frac{1}{|\mathbf{P}|}, \qquad -\frac{\partial\chi}{\partial P} = -\frac{m_1 m_2 c^3 t}{M_0}\frac{\mathbf{P}}{|\mathbf{P}|^3}, \qquad (56)$$

$$+\frac{\partial\chi}{c\,\partial t} = -\frac{m_1 m_2 c^2}{M_0}\frac{1}{|\mathbf{P}|}, \qquad (56')$$

where we have dropped the suffix 1. Writing now M for m_2, the equations of motion of the attracted particle m become

$$\frac{\xi^{\frac12}}{Y^{\frac12}}\frac{d}{dt}\left(\frac{\mathbf{V}}{Y^{\frac12}}\right) = -\frac{\xi^{\frac12}}{X}\left(\mathbf{P} - \mathbf{V}\frac{Z}{Y}\right) - \frac{Mc^3 t}{M_0}\frac{\mathbf{P}}{|\mathbf{P}|^3} + \frac{\mathbf{V}}{Y^{\frac12}}\frac{1}{Y^{\frac12}}\frac{d\xi^{\frac12}}{dt}, \qquad (57)$$

$$\frac{\xi^{\frac12}}{Y^{\frac12}}\frac{d}{dt}\left(\frac{c}{Y^{\frac12}}\right) = -\frac{\xi^{\frac12}}{X}\left(ct - c\frac{Z}{Y}\right) - \frac{Mc^2}{M_0}\frac{1}{|\mathbf{P}|} + \frac{c}{Y^{\frac12}}\frac{1}{Y^{\frac12}}\frac{d}{dt}\xi^{\frac12}. \qquad (57')$$

In these equations t refers to the epoch at the attracted particle m, and there is no longer any need to introduce the simultaneity condition (52).

To obtain the energy-integral, multiply (57) scalarly by $\mathbf{V}$, and (57') by c, and subtract. Then since

$$\frac{d}{dt}\left(\frac{t}{|\mathbf{P}|}\right) = \frac{\partial}{\partial t}\left(\frac{t}{|\mathbf{P}|}\right) + \mathbf{V}\cdot\frac{\partial}{\partial\mathbf{P}}\left(\frac{t}{|\mathbf{P}|}\right) = \frac{1}{|\mathbf{P}|} - t\frac{\mathbf{V}\cdot\mathbf{P}}{|\mathbf{P}|^3},$$

the result of this operation is

$$-\frac{d}{dt}c^2\xi^{\frac12} + \frac{Mc^3}{M_0}\frac{d}{dt}\left(\frac{t}{|\mathbf{P}|}\right) = 0,$$

or
$$c^2\xi^{\frac12} - \frac{Mc^3 t}{M_0}\frac{1}{|\mathbf{P}|} = \text{const.} \qquad (58)$$

To obtain the integral of angular momentum, multiply (57) vectorially by $\mathbf{P}$. We find

$$\frac{\xi^{\frac12}}{Y^{\frac12}}\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}}\right) = \frac{\xi}{X^{\frac12}}\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}} + \frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}}\frac{1}{Y^{\frac12}}\frac{d\xi^{\frac12}}{dt},$$

or
$$\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}}\right) = \frac{Z}{X}\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}} + \frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac12}}\frac{1}{\xi^{\frac12}}\frac{d\xi}{dt}.$$

The integrating factor is $X^{-\frac{1}{2}}\xi^{-\frac{1}{2}}$ and we get

$$\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{\xi^{\frac{1}{2}}X^{\frac{1}{2}}Y^{\frac{1}{2}}}\right)=0,$$

or integrating $\mathbf{P}\wedge\mathbf{V}=\mathbf{A}Z,$ (59)

where $\mathbf{A}$ is a vector constant.

We shall transform these to the more familiar τ-measure in a later chapter. For the moment we are content to notice that, according to (58), it is the rest-mass m that must be used in calculating the gravitational energy in the field of M. And according to (59), since $\mathbf{P}$ is always perpendicular to a fixed vector $\mathbf{A}$, the orbit lies in a plane. Further, (59) gives the usual secular increase of angular momentum with epoch, obtained previously (in t-measure). This gives a general explanation of the rotation of galaxies. The principal application of the equations of this section is in fact to the motion of a free particle in the presence of a nebular nucleus of mass M. Equations (58) and (59) contain the key to the generally spiral character of galaxies.

168. Necessity for simultaneity convention in many-body problems.

We have seen that in order that the equations of motion of n gravitating particles shall have a meaning, it is necessary to connect the epochs $t_1, t_2,\dots$ at the various particles by the invariant relations

$$X_1 = X_2 = \dots = X_n.\tag{60}$$

The various epochs $t_1, t_2,\dots$ can then be expressed in terms of a single variable, and the values of $\chi_1, m_1\xi_1^{\frac{1}{2}},\dots$ then become definite. It is important to notice that the necessity for imposing some convention as to simultaneity is not peculiar to the present theory. Any treatment of a system of n bodies which uses the ideas of conservation of energy, or of linear or angular momentum, is bound to introduce some such convention. For the principle of the conservation of energy, for example, only has a meaning when some rule is given for picking out the instants at which the energies of the various particles are to be evaluated and added together. When the energies of the various particles are varying in time, it is impossible to attach numerical values to the total energy unless the energies of the various particles can be calculated separately, and the values attributed to these will depend on the instants chosen for calculating them. The same applies to linear momentum and angular momentum. Thus

some rule is necessary for picking out corresponding instants at the separate particles, and the numerical value of the energy, linear momentum, etc., will depend on the convention chosen. The advantage of the rule (60) above is that the result of applying the rule is independent of the particular fundamental observer who applies the rule.

THE LAW OF GRAVITATION IN τ-MEASURE

169. Transformation of the gravitational potential χ. The gravitational potential χ of two particles, being an energy, is a time-invariant, and takes the same numerical value in τ-measure as in t-measure. The exact transformation of coordinates from t-measure to τ-measure being

$$l = t_0 e^{(\tau - t_0)/t_0} \cosh \lambda/ct_0, \tag{1}$$

$$l/c = t_0 e^{(\tau - t_0)/t_0} \sinh \lambda/ct_0, \tag{1'}$$

we put in our case

$$P_1 = l_1 l_1, \qquad P_2 = l_2 l_2, \tag{2}$$

and substituting for P_1, P_2, t_1, t_2 according to (1), (1') we have

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{\cosh \dfrac{\lambda_1}{ct_0} \cosh \dfrac{\lambda_2}{ct_0} - \sinh \dfrac{\lambda_1}{ct_0} \sinh \dfrac{\lambda_2}{ct_0} (l_1 . l_2)}{\left\{ \left(\cosh \dfrac{\lambda_1}{ct_0} \cosh \dfrac{\lambda_2}{ct_0} - \sinh \dfrac{\lambda_1}{ct_0} \sinh \dfrac{\lambda_2}{ct_0} l_1 . l_2 \right)^2 - 1 \right\}^{\frac{1}{2}}}. \tag{3}$$

This may be written alternatively as

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \times$$

$$\times \frac{\cosh \dfrac{\lambda_1}{ct_0} \cosh \dfrac{\lambda_2}{ct_0} - \sinh \dfrac{\lambda_1}{ct_0} \sinh \dfrac{\lambda_2}{ct_0} (l_1 . l_2)}{\left\{ \left(\cosh \dfrac{\lambda_1}{ct_0} \sinh \dfrac{\lambda_2}{ct_0} l_2 - \cosh \dfrac{\lambda_2}{ct_0} \sinh \dfrac{\lambda_1}{ct_0} l_1 \right)^2 - \sinh^2 \dfrac{\lambda_1}{ct_0} \sinh^2 \dfrac{\lambda_2}{ct_0} (l_1 \wedge l_2)^2 \right\}^{\frac{1}{2}}}. \tag{3'}$$

Formulae (3) and (3') each give the potential energy of a pair of particles, of masses m_1 and m_2, at positions (λ_1, l_1), (λ_2, l_2) respectively.

170. Independence of time-coordinates. It will be noticed that in (3) or (3') mention of the epochs τ_1, τ_2 at the two particles has completely disappeared; χ is now a function of the spatial coordinates of P_1 and P_2 only. This is of course a particular case of the general theorem ((55), Chap. VII) that since a potential function χ is homogeneous and of degree zero in the coordinates ct, P, when it is transformed to τ-measure by substitutions (1), (1') the coordinate τ drops out. Thus in τ-measure it is not necessary to specify the epoch to which χ refers.

171. Physical meaning of the formula for χ in τ-measure. To see the physical meaning of (3) or (3'), choose the origin O, to be on the line $P_1 P_2$ in the sense $O P_1 P_2$. Then $l_1 = l_2$ and $\lambda_2 > \lambda_1$. Formula (8) becomes, since $l_1 . l_2$ now equals unity,

$$\chi = -\frac{m_1 m_2 c^2}{M_0} \frac{1}{\tanh(\lambda_2 - \lambda_1)/c t_0}. \tag{4}$$

Putting γ_0 for the present value of γ, namely

$$\gamma_0 = c^3 t_0 / M_0,$$

this may be written

$$\chi = -\frac{\gamma_0 m_1 m_2}{c t_0 \tanh(\lambda_2 - \lambda_1)/c t_0}. \tag{5}$$

This value for χ is easily verified to be independent of the position of the origin O, since $\lambda_2 - \lambda_1$ is independent of O. For distances not of an order comparable with the radius $c t_0$ of the universe, so that $\lambda_2 - \lambda_1 \ll c t_0$, (5) reduces to

$$\chi = -\frac{\gamma_0 m_1 m_2}{\lambda_2 - \lambda_1}. \tag{6}$$

This agrees with the empirical Newtonian potential derived from the inverse square law. Our exact formula for the potential in the hyperbolic space of the τ-description of the substratum is, however, (5), which contains mention of t_0, the present age of the universe, over and above its occurrence in γ_0.

As $\lambda_2 - \lambda_1 \to \infty$, (5) gives

$$\chi \to -\frac{\gamma_0 m_1 m_2}{c t_0} = -\frac{m_1 m_2 c^2}{M_0}; \tag{7}$$

we obtained the same result in t-measure previously (§ 155, equation (19)).

The 'attraction' exerted by m_2 on m_1 is given by

$$-\frac{\partial \chi}{\partial \lambda_1} = \frac{m_1 m_2 c^2}{c t_0 M_0} \frac{1}{\sinh^2(\lambda_2 - \lambda_1)/c t_0} = \frac{\gamma_0 m_1 m_2}{(c t_0)^2 \sinh^2(\lambda_2 - \lambda_1)/c t_0}, \tag{8}$$

or, for $\lambda_2 - \lambda_1 \ll c t_0$,

$$-\frac{\partial \chi}{\partial \lambda_1} = \frac{\gamma_0 m_1 m_2}{(\lambda_2 - \lambda_1)^2}. \tag{9}$$

This is the inverse square law in τ-measure. The attraction tends to zero as $\lambda_2 - \lambda_1 \to \infty$. We notice further that

$$\left(-\frac{\partial \chi}{\partial \lambda_1}\right) + \left(-\frac{\partial \chi}{\partial \lambda_2}\right) = 0,$$

so that the law of equality of action and reaction holds exactly in
τ-measure, for gravitational forces.

172. Since χ is explicitly independent of epochs τ_1 and τ_2, it satisfies
Laplace's equation in the hyperbolic space $d\epsilon^2$. If we put

$$\mathbf{l}_1 = (\cos\theta_1, \sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1),$$

we can verify that

$$\frac{\partial}{\partial\lambda_1}\left((ct_0)^2\sinh^2\frac{\lambda_1}{ct_0}\frac{\partial\chi}{\partial\lambda_1}\right) + \frac{1}{\sin\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin\theta_1\frac{\partial\chi}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}\frac{\partial^2\chi}{\partial\phi_1^2} = 0, \quad (10)$$

with a similar equation in symbols suffixed 2. Thus whilst χ satisfies
a wave-equation in t-measure, it satisfies simply the equation of
Laplace in τ-measure.

173. Equations of motion under gravitation in τ-measure. By
equation (68), Chapter VII, the vector equations of motion of
particles m_1, m_2 at $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$ are in τ-measure

$$\frac{1}{(1-\mathbf{v}_1^2/c^2)^{\frac{3}{2}}}m_1\frac{d\mathbf{v}_1}{d\tau} = -\frac{\partial\chi}{\partial\mathbf{\Pi}_1}, \tag{11}$$

$$\frac{1}{(1-\mathbf{v}_2^2/c^2)^{\frac{3}{2}}}m_2\frac{d\mathbf{v}_2}{d\tau_2} = -\frac{\partial\chi}{\partial\mathbf{\Pi}_2}, \tag{11'}$$

where to a sufficient approximation

$$\chi = -\frac{\gamma_0 m_1 m_2}{|\mathbf{\Pi}_2 - \mathbf{\Pi}_1|}. \tag{12}$$

Thus the equations of motion become

$$\frac{1}{(1-\mathbf{v}_1^2/c^2)^{\frac{3}{2}}}m_1\frac{d\mathbf{v}_1}{d\tau} = -\gamma_0 m_1 m_2\frac{\mathbf{\Pi}_1-\mathbf{\Pi}_2}{|\mathbf{\Pi}_2-\mathbf{\Pi}_1|^3}, \tag{13}$$

$$\frac{1}{(1-\mathbf{v}_2^2/c^2)^{\frac{3}{2}}}m_2\frac{d\mathbf{v}_2}{d\tau} = -\gamma_0 m_1 m_2\frac{\mathbf{\Pi}_2-\mathbf{\Pi}_1}{|\mathbf{\Pi}_2-\mathbf{\Pi}_1|^3}. \tag{13'}$$

These possess the energy integral

$$\frac{m_1 c^2}{(1-\mathbf{v}_1^2/c^2)^{\frac{1}{2}}} + \frac{m_2 c^2}{(1-\mathbf{v}_2^2/c^2)^{\frac{1}{2}}} - \frac{\gamma_0 m_1 m_2}{|\mathbf{\Pi}_2-\mathbf{\Pi}_1|} = \text{const.}, \tag{14}$$

the linear momentum relation

$$\frac{m_1\, d\mathbf{v}_1/d\tau}{(1-\mathbf{v}_1^2/c^2)^{\frac{3}{2}}} + \frac{m_2\, d\mathbf{v}_2/d\tau}{(1-\mathbf{v}_2^2/c^2)^{\frac{3}{2}}} = 0, \tag{15}$$

and the angular momentum relation

$$\frac{m_1}{(1-\mathbf{v}_1^2/c^2)^{\frac{3}{2}}}\frac{d}{d\tau}(\mathbf{\Pi}_1\wedge\mathbf{v}_1) + \frac{m_2}{(1-\mathbf{v}_2^2/c^2)^{\frac{3}{2}}}\frac{d}{d\tau}(\mathbf{\Pi}_2\wedge\mathbf{v}_2) = 0, \tag{16}$$

but the last two do not appear to be integrable in terms of linear or angular momentum only, in general, unless the particles are in relative motion along a straight line.

For, in general, we have

$$\frac{d}{d\tau}\frac{\mathbf{v}_1}{(1-\mathbf{v}_1^2/c^2)^{\frac{1}{2}}} = \frac{d\mathbf{v}_1/d\tau}{(1-\mathbf{v}_1^2/c^2)^{\frac{1}{2}}} + \frac{1}{c^2}\frac{(d\mathbf{v}_1/d\tau \wedge \mathbf{v}_1) \wedge \mathbf{v}_1}{(1-\mathbf{v}_1^2/c^2)^{\frac{1}{2}}},$$

so that if L denotes the linear momentum in τ-measure in Einstein's sense, namely

$$\mathbf{L} = \sum_{1,2}\frac{m_1\mathbf{v}_1}{(1-\mathbf{v}_1^2/c^2)^{\frac{1}{2}}},$$

then (15) gives

$$\frac{d\mathbf{L}}{d\tau} = \frac{1}{c^2}\sum m_1\frac{(d\mathbf{v}_1/d\tau \wedge \mathbf{v}_1) \wedge \mathbf{v}_1}{(1-\mathbf{v}_1^2/c^2)^{\frac{3}{2}}};$$

the term on the right-hand side only vanishes if $\mathbf{v}_1$ is parallel to $d\mathbf{v}_1/d\tau$, $\mathbf{v}_2$ to $d\mathbf{v}_2/d\tau$, and this will occur only if the velocities $\mathbf{v}_1$, $\mathbf{v}_2$ are along the line joining the particles. It must be remembered that the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are measured relative to the local standard of rest. The formulae containing them are not therefore necessarily applicable to the general problem of the motion of any two bodies relative to their centre of mass, but only to motions relative to a nebular nucleus.

174. Alternative derivation. It can be shown that relations (15) and (16) also follow from the exact relations (43′) and (47′) of the preceding chapter. For as usual

$$\xi_1^1 = \frac{Z_1}{X_1^1 Y_1^1} = \frac{1}{Y_1^1}\frac{dX_1^1}{dt} = \frac{t_0 d(e^{(\tau-t_0)/t_0})}{e^{(\tau-t_0)/t_0}d\sigma} = \frac{d\tau}{d\sigma} = \left(1-\frac{\mathbf{v}_1^1}{c^2}\right)^{-\frac{1}{2}},$$

and $$\frac{X_1^1}{Y_1^1}\frac{d}{dt_1} = \frac{t_0 e^{(\tau-t_0)/t_0}}{e^{(\tau-t_0)/t_0}}\frac{d}{d\sigma_1} = \frac{d\tau_1}{d\sigma_1}t_0\frac{d}{d\tau_1} = \left(1-\frac{\mathbf{v}_1^1}{c^2}\right)^{-\frac{1}{2}}t_0\frac{d}{d\tau_1}.$$

Hence (43′) or (54) yields, on putting $\tau_1 = \tau_2 = \ldots = \tau$,

$$\sum_r m_r\left(1-\frac{\mathbf{v}_r^2}{c^2}\right)^{-\frac{3}{2}}\frac{d\mathbf{v}_r}{d\tau} = 0,$$

and (47′) or (55) yields

$$\sum_r m_r\left(1-\frac{\mathbf{v}_r^2}{c^2}\right)^{-\frac{3}{2}}\frac{d}{d\tau}(\Pi_r \wedge \mathbf{v}_r) = 0.$$

175. General inference. It has previously been remarked that the equations of motion to which we have been led do not coincide exactly with Einstein's 'special relativity' equations of motion. The latter

depend on a confusion of ideas concerning the time-variable, the same variable being employed in Lorentz-transformation theory and in the equations of motion of pseudo-Newtonian form. In our work, on the other hand, when we use the Lorentz-transformation theory, in consequence of our adoption of a scale of time t in which the fundamental observers or fundamental frames of reference are in uniform relative motion, the equations of motion have been of non-Newtonian form, and have only reduced to a form similar to, though not identical with, Einstein's, when we have made a logarithmic transformation of the time-variable. It is in consequence of these differences that we have not been led to any exact integrals of linear or angular momentum in the many-particle problem. On the other hand, we have had exact integrals of energy, these being connected with the circumstance that, in the present theory, energy is an invariant, and not the time-component of a 4-vector.

The general inference from these investigations is that doubt must be cast on the validity of any world-wide principles of linear momentum or angular momentum for unrestricted velocities, though they will hold very closely for velocities not comparable with c. Also they hold strictly for collinear sets of particles.

THE STRUCTURE OF A SPIRAL NEBULA

176. The Keplerian problem in τ-time. We propose as an example in gravitational theory of deep cosmical interest to apply our equations of motion to determine the orbits of a set of particles in the neighbourhood of a massive particle M, and to investigate the light these orbits shed on the structure of a spiral nebula. We shall consider these orbits in both t- and τ-measure, but it must always be borne in mind that t-measure is the more fundamental, as not containing the normalization constant t_0. We shall see that the analysis is intimately dependent on a correct interpretation of constants such as t_0, and on their elimination.

We shall take the particle M to be a fundamental particle, and to represent the nucleus of an extra-galactic nebula. It is recognized that the nucleus of an extra-galactic nebula is something more complicated than a particle, but this idealization may be permitted in a first assault on this problem. It will not be assumed that the spiral arms of a nebula represent orbits. Instead we shall take the arms for what they are—loci of the present positions of the particles constituting the outer parts of the nebula.

For $|\mathbf{P}| \ll ct$, $|\mathbf{V}| \ll c$, the t-equation of motion in the Keplerian problem, namely equation (57) of Chapter X, takes the approximate form

$$\frac{d\mathbf{V}}{dt} = -\frac{\mathbf{P} - \mathbf{V}t}{t^2} - \frac{Mc^3t}{M_0}\frac{\mathbf{P}}{|\mathbf{P}|^3}. \tag{1}$$

We shall confine attention to motion in one plane, and therefore (1) is equivalent to two scalar equations. We shall take one of these to be the equation giving the radial component of acceleration in plane polar coordinates, namely

$$\ddot{r} - r\dot{\theta}^2 = -\frac{r - \dot{r}t}{t^2} - \frac{Mc^3t}{M_0}\frac{1}{r^2}. \tag{2}$$

The other we shall take in its integrated form (59), Chapter X, namely

$$r^2\dot{\theta} = \text{const.} \times t. \tag{3}$$

In (2) and (3), dots denote differentiation with respect to t.

To solve these equations, we transform to τ-measure.

· We put, as usual

$$\frac{dt}{t} = \frac{d\tau}{t_0}, \qquad \tau = t_0 \log \frac{t}{t_0} + t_0,$$ (4)

and set

$$r = \frac{t}{t_0}\rho,$$ (5)

Then

$$\dot{r} = \frac{\rho}{t_0} + \frac{t}{t_0}\frac{d\rho}{dt} = \frac{r}{t} + \frac{d\rho}{d\tau},$$

and

$$\ddot{r} = \frac{\dot{r}}{t} - \frac{r}{t^2} + \frac{t_0}{t}\frac{d^2\rho}{d\tau^2}.$$

Hence (2) becomes

$$\frac{t_0}{t}\frac{d^2\rho}{d\tau^2} - \frac{t}{t_0}\rho\left(\frac{t_0}{t}\frac{d\theta}{d\tau}\right)^2 = -\frac{Mc^3t}{M_0}\left(\frac{t_0}{t}\right)^2\frac{1}{\rho^2},$$

or, dividing through by the factor t_0/t,

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{d\theta}{d\tau}\right)^2 = -\frac{Mc^3t_0}{M_0}\frac{1}{\rho^3} = -\frac{\gamma_0 M}{\rho^2}.$$ (6)

Similarly the angular momentum integral (3) becomes

$$\left(\frac{t}{t_0}\right)^2\rho^2\left(\frac{t_0}{t}\right)\frac{d\theta}{d\tau} = \text{const.} \times t,$$

or

$$\rho^2\frac{d\theta}{d\tau} = \text{const.} \times t_0.$$ (7)

We see that as usual the transformation (4), (5) removes the cosmical acceleration term from the equation of motion, and transforms the gravitational coefficient γ_t given by

$$\gamma_t = \frac{c^3 t}{M_0},$$

into a constant γ_0, given by

$$\gamma_0 = \frac{c^3 t_0}{M_0}.$$

Moreover, the same transformation removes the secular proportionality of the angular momentum to t, leaving it constant, though proportional to the constant t_0. We recognize (6) and (7) as the usual Keplerian equations with ρ, τ as distance- and time-variables.

177. Limitation to rotation. The general solution of equations (6) and (7) will contain three parameters: a scale parameter, a shape parameter, and an orientation parameter. The shape parameter

may conveniently be taken to be the eccentricity e. But non-zero values of e will correspond to orbits which give rise to the phenomenon of star-streaming in the family of orbits considered. The primary phenomenon in a galaxy is *rotation*, and we shall therefore confine attention to orbits for which $e = 0$, i.e. circular orbits in τ-measure. Relation (3), or its τ-form (7), then exhibits the angular momentum of the system as arising from the secular magnification, by the couple due to the rest of the universe, of any initial amount of angular momentum resident in the system.

178. If the orbits in τ-measure are taken to be the circles

$$\rho = \text{const.} = \rho_0, \tag{8}$$

then (6) and (7) give

$$\frac{d\theta}{d\tau} = \text{const.}$$

The value of the constant is given by (6) in the form

$$\frac{d\theta}{d\tau} = \left(\frac{\gamma_0 M}{\rho_0^3}\right)^{\frac{1}{2}}. \tag{9}$$

This is simply the statement of Kepler's third law for our case. If at epoch t_0 the particle in its orbit is just passing through the position (ρ_0, θ_0), then (9) takes the integrated form

$$\theta - \theta_0 = \left(\frac{\gamma_0 M}{\rho_0^3}\right)^{\frac{1}{2}} (\tau - t_0). \tag{10}$$

Relations (8) and (10) give a typical circular orbit expressed in τ-measure.

The same orbit expressed in t-measure is given, on using (4) and (5), by

$$r = \rho_0 \frac{t}{t_0}, \tag{11}$$

$$\theta - \theta_0 = \left(\frac{\gamma_0 M t_0^2}{\rho_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0}. \tag{12}$$

Eliminating the time variable t, the (r, θ) equation of the orbit is

$$\theta - \theta_0 = \left(\frac{\gamma_0 M t_0^2}{\rho_0^3}\right)^{\frac{1}{2}} \log \frac{r}{\rho_0}. \tag{13}$$

This is the equation of an equi-angular spiral, of angle α given by

$$\tan \alpha = r \frac{d\theta}{dr} = \left(\frac{\gamma_0 M t_0^2}{\rho_0^3}\right)^{\frac{1}{2}}. \tag{14}$$

179. Orbits apparently re-entrant. To the observer situated at the nebular nucleus as origin, and using the expanding metre of classical physics, the orbits will appear as re-entrant circles. The orbits will conserve this property of *appearing* re-entrant, even to a distant observer on another nebular nucleus who is using the more fundamental t-measure. For in t-measure, an observer on one nucleus will reckon the distant nucleus as receding with some velocity V; he will reckon its distance as Vt; and he will reckon the angle subtended by the radius vector of the orbital particle in the plane of the distant nebula (assuming he is viewing it normally), as

$$r/Vt = \rho_0/Vt_0 = \text{const.}$$

If he plots the orbit, however, in three-dimensional space, he will consider it as non-re-entrant, for it will take the form of a three-dimensional spiral lying on the surface of a cone, whose vertex is at himself and whose semi-vertical angle is ρ_0/Vt_0.

We do not, however, see the actual orbits of the particles constituting a nebula, plotted in the sky. We see the *present positions* (ρ_0, θ_0) of the particles, each in its own orbit. The appearance of a spiral nebula in the sky suggests that we are viewing the present positions of the particles in a one-parameter family of orbits. If this is so, there must be some relation between the two independent parameters θ_0 and ρ_0 which occur in equations (11) and (12), or (8) and (10). This relation, say $\theta_0 = f(\rho_0)$, which will reduce the family of orbits to a one-parameter family, will be precisely the equation of the locus of the present positions of the particles, which is what we wish to determine. It will be the equation of the present position of the spiral arms.

180. The envelope of the orbits. If the orbits form a one-parameter family, they may possess an envelope. It was suggested by E. W. Brown[†] that the observed arms of spiral nebulae are constituted by such an envelope. But there are two arguments against this view. One is that it is highly unlikely that the majority of the particles would be passing through the points of contact of their orbits with the envelope at the moment we happen to be viewing the system. The other is that the orbits will be relatively closely crowded together near their envelope, and would be likely to perturb one another, and

† *Astrophys. Journ.* 61, 111, 1925.

the resulting appearance would be unlikely to be that of a one-parameter family of orbits, which is our hypothesis.

For if the relation between the two parameters θ_0 and ρ_0 is as before

$$\theta_0 = f(\rho_0),$$

and if the actual orbits of the one-parameter family are in (r, θ) coordinates

$$F(r, \theta, \rho_0) = 0,$$

then the envelope is obtained by eliminating ρ_0 between the latter relation and

$$\frac{\partial F}{\partial \rho_0} = 0.$$

Suppose now that there are $n(\rho_0)\, d\rho_0$ orbits (i.e. particles) with parameters between ρ_0 and $\rho_0 + d\rho_0$ distributed along the present positions $\theta_0 = f(\rho_0)$. Then when the radius vector of a given particle is r, the orbit-density in r is given by $N(r)\, dr$, where

$$N(r)\, dr = n(\rho_0)\, d\rho_0.$$

But dr and $d\rho_0$ are connected by

$$\frac{\partial F}{\partial r}\, dr + \frac{\partial F}{\partial \rho_0}\, d\rho_0 = 0.$$

Hence

$$N(r) = -\frac{n(\rho_0)\, \partial F/\partial r}{\partial F/\partial \rho_0},$$

which at a point of contact with the envelope becomes indefinitely large.

181. Condition of absence of envelope. It becomes clear, in fact, from the regularity of pattern of a galaxy, that the orbits cannot have had an envelope. We shall use this as a clue to give the distribution of orbits.

We must first emphasize that t-measure is the more fundamental. As it is essential to recognize that the galaxies are receding, we use the equations of the orbits in t-measure. To emphasize that we are using t-measure, we shall replace ρ_0 by r_0. The equation of the orbits is then, by (13),

$$\theta - f(r_0) = \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{r}{r_0}.$$

These orbits, with parameter r_0, will have an envelope obtained by eliminating r_0 between the latter equation and

$$-f'(r_0) = \frac{1}{r_0}\left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}}\left(-\frac{3}{2}\log\frac{r}{r_0} - 1\right).$$

The envelope will therefore be given parametrically in terms of r_0 by

$$\log \frac{r}{r_0} = -\frac{2}{3} + \frac{2}{3} r_0 f'(r_0)\left(\frac{r_0^3}{\gamma_0 M l_0^2}\right)^{\frac{1}{2}},$$

$$\theta = f(r_0) + \frac{2}{3} r_0 f'(r_0) - \frac{2}{3}\left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}.$$

The family of equi-angular spiral orbits will therefore possess an envelope unless the expression for θ degenerates. It will degenerate to a constant, say β, if $f(r_0)$ is such that identically

$$f(r_0) + \frac{2}{3} r_0 f'(r_0) - \frac{2}{3}\left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}} \equiv \beta.$$

This may be written

$$\frac{2}{3}\frac{d}{dr_0}\{r_0^{\frac{3}{2}} f(r_0)\} = \frac{2}{3}\frac{(\gamma_0 M l_0^2)^{\frac{1}{2}}}{r_0} + \beta r_0^{\frac{1}{2}}.$$

This integrates in the form

$$r_0^{\frac{3}{2}} f(r_0) = \beta r_0^{\frac{3}{2}} + (\gamma_0 M l_0^2)^{\frac{1}{2}}\log r_0 + \text{const.},$$

or

$$f(r_0) = \beta + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log r_0 + \frac{\text{const}}{r_0^{\frac{3}{2}}}.$$

This can be written without loss of generality in the form

$$f(r_0) = \beta + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log \frac{r_0}{r_1},$$

where r_1 is some constant.

Thus the envelope degenerates to the line $\theta = \beta$ when the present positions of the particles in their orbits are given by

$$\theta_0 = \beta + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log \frac{r_0}{r_1}. \tag{15}$$

182. If this equation gives the present positions of the particles (r_0, θ_0) in their orbits, then the equation of an orbit in t-measure takes the parametric form (eliminating θ_0 from (12)),

$$\theta = \beta + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log \frac{r_0}{r_1} + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log \frac{l}{l_0}, \qquad r = r_0 \frac{l}{l_0}. \tag{10}$$

The (r, θ) equation of an orbit which reaches the radial distance r_0 at time t_0 is accordingly

$$\theta = \beta + \left(\frac{\gamma_0 M l_0^2}{r_0^3}\right)^{\frac{1}{2}}\log \frac{r}{r_1}. \tag{17}$$

Hence all orbits pass through the fixed point (in t-measure)

$$r = r_1, \qquad \theta = \beta.$$

The locus (15) also passes through this point, which is readily found to be the point of contact of each orbit with the envelope. Thus the degeneracy consists in the envelope reducing to a single point.

183. Physical description of the family of orbits. The family of orbits to which we have been led consists of orbits pursued by the different particles which have successively passed through the fixed point $r = r_1$, $\theta = \beta$. Now it has been a feature of theories of nebular evolution to regard the arms as consisting of particles *emitted* from one, two, or more points on the rim of a nebular nucleus. Without having had these theories in mind, we have been led to consider the family of orbits represented in a spiral nebula as precisely a family of orbits pursued by particles emitted successively from a fixed point.

Let us now adopt this as an hypothesis, and explore the consequences. The orbit (11), (12), will have passed through the fixed point (β, r_1) at the epoch t_1 if

$$\beta - \theta_0 = \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{t_1}{t_0}, \qquad r_1 = r_0 \frac{t_1}{t_0}. \qquad (18),\ (19)$$

Eliminating t_1 between these equations, we have

$$\theta_0 = \beta + \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{r_0}{r_1}, \qquad (20)$$

which is precisely the locus (15). Thus (20) gives the equation to the spiral arm at epoch t_0, being the relation connecting θ_0 and r_0, the position at time t_0. Let us see what this locus becomes at time t.

The particle at (r_0, θ_0) at time t_0 passes to (r, θ) at time t, where

$$\theta - \theta_0 = \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0}, \qquad r = r_0 \frac{t}{t_0}. \qquad (21),\ (22)$$

Eliminate r_0 and θ_0 between (20), (21), and (22). Then the (r, θ) locus at time t is given by

$$\theta = \beta + \left(\frac{\gamma_0 M t_0^2 t^3}{r^3 t_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0} + \left(\frac{\gamma_0 M t_0^2 t^3}{r^3 t_0^3}\right)^{\frac{1}{2}} \log \frac{r t_0}{r_1 t}; \qquad (23)$$

but

$$\frac{\gamma_0 M t_0^2 t^3}{r^3 t_0^3} = \frac{\gamma_t M t^2}{r^3},$$

on using $\gamma_0 = c^3 t_0/M_0$, $\gamma_t = c^3 t/M_0$.   Hence (23) becomes

$$\theta = \beta + \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log \frac{r}{r_1}. \tag{24}$$

This should be the equation of the spiral arm at time t, assuming (20) gives the spiral arm at time t_0. We now note the remarkable fact that in the course of eliminating θ_0 and r_0 from (20) by means of the parametric equations of the orbit, (21) and (22), we have also eliminated t_0. Locus (24) contains no mention of t_0. Further, (24) is precisely the form which (20) would take if we replaced t_0 wherever it occurs in (20) by t.

Thus (24) represents the permanent equation of a spiral arm, valid at any time t. It contains besides the constants β and r_1 and the nuclear mass M, only the epoch t; t_0 is not present. We see that this result depends intimately on the 'constant' of gravitation being proportional to t.

184. The permanent equation of a spiral arm. Let us now see whether (24) is the most general equation possessing these properties. Suppose, quite generally, that the equation of a spiral arm at time t is

$$\theta = F(r, t), \tag{25}$$

so that at time t_0 it is

$$\theta_0 = F(r_0, t_0). \tag{26}$$

Then as the particles are displaced along their orbits from time t_0 to time t, (26) must pass into (25), and so, using (21) and (22),

$$F(r, t) - F(r_0, t_0) = \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0}.$$

Eliminating r_0 by means of (22), we get

$$F(r, t) - F\left(\frac{rt_0}{t}, t_0\right) = \left(\frac{\gamma_0 M t_0^2 t^3}{r^3 t_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0} = \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log \frac{t}{t_0}. \tag{27}$$

Put

$$F(r, t) = \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log \frac{r}{r_1} + \psi(r, t),$$

where r_1 is some constant.   Then

$$F\left(\frac{rt_0}{t}, t_0\right) = \left(\frac{\gamma_0 M t_0^2 t^3}{r^3 t_0^3}\right)^{\frac{1}{2}} \log \frac{rt_0}{r_1 t} + \psi\left(\frac{rt_0}{t}, t_0\right).$$

Inserting these in (27) we get

$$\psi(r, t) - \psi\left(\frac{rt_0}{t}, t_0\right) = 0.$$

Put
$$\frac{r}{t} = x.$$

Then
$$\psi(xt, t) - \psi(xt_0, t_0) = 0.$$

Thus $\psi(xt, t)$ is a function of x only, say
$$\psi(xt, t) = \Phi(x),$$
or
$$\psi(r, t) = \Phi(r/t).$$

Thus the most general form of an arm of a spiral, that is, of the present positions of the particles, when the orbits are the equi-angular spirals (21), (22), is

$$\theta = \Phi\left(\frac{r}{t}\right) + \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log\frac{r}{r_1}.$$

But the argument of the function Φ, as well as Φ itself, must be of zero physical dimensions. We therefore need to divide r/t by a constant of the dimensions of a velocity. Seeking as we are a *general* theory of spirals, we have no such constant available except c, the speed of light. We should therefore expect the form of an arm to be given by

$$\theta = \Phi\left(\frac{r}{ct}\right) + \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log\frac{r}{r_1}.$$

But
$$\frac{r}{ct} \sim 0$$

whence
$$\Phi\left(\frac{r}{ct}\right) \sim \text{const.} = \beta,$$

say. Hence, substantially, the most general equation of a spiral arm must be of the form

$$\theta = \beta + \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log\frac{r}{r_1},$$

which is precisely (24).

It is possible, of course, by suitable arrangement of the particles constituting a nebula at some time t_0, to have $\Phi(r_0/t_0)$ an arbitrary function of r_0, say $S(r_0)$. Then $\Phi(r/t) = S(rt_0/t)$, but this will not be in general independent of t_0. It is in this sense that (24) may be described as the permanent equation of a nebular arm.

185. Alternative form of equation of orbit. We have given already two equivalent forms of the equation of an orbit, namely

$$\theta = \theta_0 + \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log\frac{t}{t_0}, \qquad r = r_0\frac{t}{t_0}, \qquad (28), (29)$$

and
$$\theta = \beta + \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_1}, \qquad r = r_1 \frac{t}{t_1}. \tag{30}, (31)$$

Since
$$r_0 = r_1 \frac{t_0}{t_1},$$

equation (30) may also be put in the form

$$\theta = \beta + \left(\frac{\gamma_0 M t_0^3 t_1^3}{r_1^3 t_0^3}\right)^{\frac{1}{2}} \log \frac{t}{t_1}$$

$$= \beta + \left(\frac{\gamma_1 M t_1^2}{r_1^3}\right)^{\frac{1}{2}} \log \frac{t}{t_1}, \tag{32}$$

where
$$\gamma_1 = c^3 t_1 / M_0.$$

This gives the orbit of the particle which has passed through the fixed point (r_1, β) at epoch t, in terms of the constants r_1 and β and the parameter t_1. The relation of this form of the (r, θ) equation of the orbit, namely

$$\theta = \beta + \left(\frac{\gamma_1 M t_1^2}{r_1^3}\right) \log \frac{r}{r_1},$$

to the (r, θ) equation of the arm at any time t, namely (24), should be noted.

186. Properties of the spiral arm. The orbits, we have seen, are in t-measure equi-angular spirals. The equation (24) we have found for the arm of a nebula is also a spiral, but not an equi-angular one. It has the unexpected property that it makes a finite number of turns in the sense of the orbital motion, and then an equal number of turns in the opposite sense.

For, from (24), in the usual notation,

$$\tan \phi = r \frac{d\theta}{dr} = \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \left(1 - \frac{3}{2} \log \frac{r}{r_1}\right).$$

Thus $\tan \phi$ passes through a zero at $r = e^{\frac{1}{3}} r_1 = 1 \cdot 0477 \, r_1$, and then changes sign. It is clear, in fact, that $\theta \to \beta$ for $r \to r_1$ and for $r \to \infty$, and that in between it has a single maximum, namely at $r = e^{\frac{1}{3}} r_1$. The number of turns made between $\theta = \beta$ and $\theta = \theta_{max}$ is say ν, given by

$$\nu = \frac{1}{2\pi} [\theta_{max} - \beta] = \frac{1}{2\pi} \left(\frac{\gamma_t M t^2}{e^2 r_1^3}\right)^{\frac{1}{2}} \frac{2}{3} = \frac{1}{3\pi e} \left(\frac{\gamma_t M t^2}{r_1^3}\right)^{\frac{1}{2}}. \tag{33}$$

The equation of the nebula arm can therefore be written

$$\theta = 3\pi e \nu \left(\frac{r_1}{r}\right)^{\frac{3}{2}} \log \frac{r}{r_1}. \tag{34}$$

At epoch t_0, $$\nu = \nu_0 = \frac{1}{3\pi e}\left(\frac{\gamma_0 M t_0^2}{r_1^3}\right)^{\frac{1}{2}} \tag{35}$$

and $$\frac{\nu}{\nu_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{36}$$

It is clear that r_1 is the scale parameter, ν the shape parameter, of the spiral arm. We have, moreover,

$$\nu_0 = \frac{1}{3\pi e}\left(\frac{c^3 t_0^3}{M_0}\frac{M}{r_1^3}\right)^{\frac{1}{2}}, \tag{37}$$

so that r_0 is proportional to the square root of the ratio of the mean density of the nuclear region of the nebula to the present mean density of the universe near ourselves.

For a galaxy similar to our own galaxy, with M/r_1^3 corresponding to the mean density inside the sun's distance from the centre, the order of magnitude of ν_0 as given by (35) is about 2·0. This is in agreement with the observed fact that the number of convolutions in spiral nebulae is generally of the order of 2 or 3. But more important still is the circumstance that actual reversals of the sense of turning of spiral arms have been observed by Lindblad† for the spirals NGC 2681 and 3190. These spirals were found by Lindblad to possess faint outer arms which *trail*, in addition to the main inner arms which proceed in the sense of the orbital rotation. The existence of a turning-point was in fact first suggested by Lindblad, on a theory different from the one here propounded.

The turning-back of the spiral arm is more sudden the larger is ν, but it is fairly sudden even for ν so small as 2. For ν as small as $\frac{1}{2}$, the general effect for a pair of arms issuing from a pair of antipodal points (r_1,β), $(r_1,\beta+\pi)$ is to give an appearance resembling a *barred* spiral.

It is not necessary to the present theory that every spiral nebula should exhibit both direct and retrograde convolutions. The material in the vicinity of the turning-point was emitted at the epoch

$$t_1 = e^{-1}t_0 = 0\cdot5134 t_0,$$

and the outer convolutions must have been emitted at epochs t_1 earlier than this value, the inner convolution at epochs t_1 later than this value. If emission of material from a point (r_1,β) began before $t_1 = 0\cdot5134 t_0$, outer retrograde convolutions should be present; if

† *Stockholm Obs. Ann.* 14, No. 3, 1942.

emission occurred after $t_1 = 0.5134t_0$, inner, direct convolutions should be present.

Lindblad has suggested that both direct and retrograde convolutions may be present in the Andromeda nebula, M 31.

187. General conclusions. Our general conclusion is that the spiral arms of extra-galactic nebulae are the loci of the present positions of particles emitted at various epochs from one or more fixed points. This is not an inevitable deduction from kinematic relativity, but it is the conclusion to which kinematic relativity points.

It was suggested by Jeans in his Adams Prize Essay of 1917, *Problems of Cosmogony and Stellar Dynamics*, that spiral arms might originate at points in the edge of a lenticular mass of compressible material in rotation, the points being determined by the general tidal action of the rest of the universe. There would of course be no net tidal action on a nucleus which was strictly a fundamental particle, as the rest of the universe would be disposed symmetrically about it; but owing to departures of the system of the galaxies from the ideal substratum, there will have been in general a residual tidal pull at the time the edge was in a state to shed material. Though Jeans accounted in this way for the *presence* of nebular arms, he was unable to explain their spiral character. In 1928 he summed up his views† by saying that

'the further interpretation (of the spiral arms) forms one of the most puzzling, as well as disconcerting, problems of cosmogony. . . . Each failure to explain the spiral arms makes it more and more difficult to resist a suspicion that the spiral nebulae are the seat of types of forces entirely unknown to us, forces which may possibly express novel and unsuspected properties of space. The type of conjecture which presents itself, somewhat insistently, is that the centres of the nebulae are of the nature of singular points at which matter is poured into our universe from some other, and entirely extraneous, spatial dimension, so that, to a denizen of our universe, they appear as points at which matter is continuously being created.'

The analysis of the present volume is in general conformity with these speculations of Jeans, though it has had a very different origin. The centre of each nebula, before the nebulae separated from one another by the expansion of the universe, has indeed been a singular point, where matter was created at some supra-sensual event, which is the origin of time for our t-scale. In my earlier monograph,

† *Astronomy and Cosmogony*, p. 352.

Relativity, Gravitation, and World-Structure, I traced the separation
of the nebular nuclei from one another, in virtue of the expansion
from the initial grand singularity, and showed how they carried
aggregations of matter with them. The statistical analysis of that
volume described a highly idealized system of idealized galaxies, and
did not go further than to endow every nebula with spherical sym-
metry. It traced the interchange of membership by particles of one
system with members of other systems. The present investigation
has completed the picture, by tracing in detail the *local* gravitational
effects on the motions of the particles belonging to each sub-system.
Assuming that each sub-system possesses a plane of symmetry and
an axis of rotation, we have shown how the angular momentum is
increased as time advances. Then, using the deductive theory of
gravitation and dynamics here developed, we have traced the evolu-
tion of a spiral arm, as the locus of present positions of particles
themselves describing spiral orbits. Chief amongst the new properties
of gravitation to whose existence Jeans speculatively appealed is the
secular dependence of the 'constant' of gravitation on the epoch.
We have seen that due to this we could isolate a permanent equation
of a nebular arm, independent of special initial conditions, an equa-
tion whose general form at time t is independent of the normalization
constant t_0. We have connected in an intimate way the spiral form
of the nebulae with their recession. (It has been repeatedly em-
phasized by Vogt that there should be a cosmological connexion
between these two phenomena.) Both effects can only be properly
treated by using t-measure of time, and not the ordinary dynamical or
Newtonian measure of time, τ. The latter is an ephemeral measure of
time, since the descriptions of phenomena in terms of τ-measures
involve mention of a parameter t_0, which plays no part in t-measure.
The spiral character of the nebulae writes for us in the heavens the
message that the 'constant' of gravitation is not truly constant, just
as the red-shift writes for us in our spectroscopes the message of the
expansion and the natural origin of time. We see why the galaxies
have a fairly common pattern. Their shapes depend on a single
parameter, measured by the ratio of the nuclear mean density to the
mean density of matter in the smoothed-out universe, this parameter
determining the number of convolutions of a spiral arm. We see in
a general way why this number of convolutions is in general small,
for it is equal to the square root of the above ratio divided by

$3\pi e = 25\cdot6$. The theory also predicts the reversal of the sense of winding of the spiral arms as possible in certain nebulae, a reversal which has been observed by Lindblad.

A most satisfactory aspect of the general theory of the spiral nebulae here put forward is that it makes no sudden break with Newtonian theory, but involves a modification of it which is only evident over long periods of time. This modification is not empirical in origin, but is essentially connected with the basic theory of measurement on which kinematic relativity relies.   The spiral character of the nebulae is in fact a substantial piece of observational evidence that kinematic relativity is on right lines.

ELECTRODYNAMICS

XIII

THE ELECTRODYNAMICS OF POINT-CHARGES

188. Hierarchy of possible forces. We have seen in Chapter X how we derived gravitational forces from the differentiation of a scalar potential χ. This referred explicitly to a pair of particles, and it had a singularity when the two particles approached coincidence. This singularity arose in connexion with a denominator $(X_{12}^2 - X_1 X_2)^{-\frac{1}{2}}$, which plays the part in kinematic relativity that the denominator $1/r$ plays in elementary potential theory. By the addition of terms of this type, or by integration, we can generate more and more complicated gravitational situations, including the gravitational properties of continuous or quasi-continuous distributions of matter, but we shall never, by this means, emerge from the domain of gravitation. First-order differentiation of 4-scalar potentials may be expected only to generate forces of gravitational character. To represent the phenomena of electrodynamics we require something essentially different.

The next hierarchy of possible forces after gravitational forces may be supposed derived by the *second-order* partial differentiation of 4-scalars which may be called super-potentials. The second-order differentiation of a 4-scalar yields, however, a tensor of the second rank, whilst what we are seeking is a 3-vector, namely, something to express the mechanical force acting on a charged particle due to the presence of an electromagnetic field. To derive a vector from a tensor, our only recourse is to form its inner product with an existing vector. The only 4-vectors at our disposal associated with a particle in motion are the position-epoch 4-vector $(\mathbf{P}, ct)$ and the velocity 4-vector $(\mathbf{V}/Y^{\frac{1}{2}}, c/Y^{\frac{1}{2}})$. We shall content ourselves with exploring the consequences of using the association of the rank-2 tensor and the velocity 4-vector to represent the new type of mechanical force.

189. Introduction of super-potentials. Take an undefined scalar ϕ_{21}, a function of two 4-vectors, $(\mathbf{P}_1, ct_1)$ and $(\mathbf{P}_2, ct_2)$, and possibly of their associated velocity-vectors. We seek to define ϕ_{21} in such

a way that its double partial differentiation is to represent the effect of a singularity at P_2 on a test-particle at P_1. It is to become infinite as $P_1 \to P_2$, when at the same time $t_1 \to t_2$. We shall write

$$T_{23} = \frac{\partial^2 \phi_{21}}{\partial y_1 \, \partial z_2} - \frac{\partial^2 \phi_{21}}{\partial z_1 \, \partial y_2}, \tag{1}$$

$$T_{31} = \frac{\partial^2 \phi_{21}}{\partial z_1 \, \partial x_2} - \frac{\partial^2 \phi_{21}}{\partial x_1 \, \partial z_2}, \tag{1'}$$

$$T_{12} = \frac{\partial^2 \phi_{21}}{\partial x_1 \, \partial y_2} - \frac{\partial^2 \phi_{21}}{\partial y_1 \, \partial x_2}, \tag{1''}$$

$$T_{14} = \frac{1}{c}\left(\frac{\partial^2 \phi_{21}}{\partial x_1 \, \partial t_2} - \frac{\partial^2 \phi_{21}}{\partial x_2 \, \partial t_1} \right), \tag{2}$$

$$T_{24} = \frac{1}{c}\left(\frac{\partial^2 \phi_{21}}{\partial y_1 \, \partial t_2} - \frac{\partial^2 \phi_{21}}{\partial y_2 \, \partial t_1} \right), \tag{2'}$$

$$T_{34} = \frac{1}{c}\left(\frac{\partial^2 \phi_{21}}{\partial z_1 \, \partial t_2} - \frac{\partial^2 \phi_{21}}{\partial z_2 \, \partial t_1} \right), \tag{2''}$$

$$T_{32} = -T_{23}, \qquad T_{13} = -T_{31}, \qquad T_{21} = -T_{12},$$

$$T_{41} = -T_{14}, \qquad T_{42} = -T_{24}, \qquad T_{43} = -T_{34},$$

$$T_{11} = 0, \qquad T_{22} = 0, \qquad T_{33} = 0, \qquad T_{44} = 0.$$

Here we have written (x_1, y_1, z_1) for P_1, (x_2, y_2, z_2) for P_2. We shall call $\mathbf{H}_1$ the 3-vector constituted by the components T_{23}, T_{31}, T_{12} of the tensor $\mathbf{T}$; and we shall call $\mathbf{E}_1$ the 3-vector constituted by the components T_{14}, T_{24}, T_{34} of $\mathbf{T}$. We do not at this stage call $\mathbf{H}_1$ the magnetic intensity at P_1 or $\mathbf{E}_1$ the electric intensity at P_1; we shall have later to identify $\mathbf{H}_1$ and $\mathbf{E}_1$ with the magnetic and electric intensities from their properties. The complete name-scheme for the components of $\mathbf{T}$ is then expressed by

$$\left.\begin{matrix} T_{11}, & T_{12}, & T_{13}, & T_{14} \\ T_{21}, & T_{22}, & T_{23}, & T_{24} \\ T_{31}, & T_{32}, & T_{33}, & T_{34} \\ T_{41}, & T_{42}, & T_{43}, & T_{44} \end{matrix}\right\} = \left\{\begin{matrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0. \end{matrix}\right.$$

$\mathbf{T}$ is a skew-symmetric covariant tensor. We call $T_{\mu\nu}$ the 6-vector $(\mathbf{H}, \mathbf{E})$. The metric for this tensor is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = c^2\, dt^2 - dx^2 - dy^2 - dz^2,$$

where we have written $dx_4 = c\, dt$. Associated with $T_{\mu\nu}$ are the

contravariant tensor $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$, the transposed covariant tensor $\mathcal{T}_{\mu\nu} = \tfrac{1}{2}A_{\mu\nu\alpha\beta}T^{\alpha\beta}$, and the transposed contravariant tensor

$$\mathcal{T}^{\mu\nu} = \tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}\mathcal{T}_{\alpha\beta},$$

A being the alternate tensor associated with ds^2.

If Q^α, R^α are contravariant 4-vectors then their space parts form 3-vectors Q, R and we shall write $Q^4 = Q_t$, $R^4 = R_t$. We can construct from Q^α and R^α the covariant skew-symmetric tensor $A_{\mu\nu\alpha\beta}Q^\alpha R^\beta$ and its contravariant and transposed forms. The complete scheme of components expressed in terms of 3-vectors and associated scalars is then

$$
\begin{aligned}
T_{\mu\nu} &= (\mathbf{H}, \mathbf{E}), & S_{\mu\nu} &= (\mathbf{Q}R_t - \mathbf{R}Q_t, \mathbf{Q} \wedge \mathbf{R}), \\
T^{\mu\nu} &= (\mathbf{H}, -\mathbf{E}), & S^{\mu\nu} &= (\mathbf{Q}R_t - \mathbf{R}Q_t, -(\mathbf{Q} \wedge \mathbf{R})), \\
\mathcal{T}_{\mu\nu} &= (-\mathbf{E}, \mathbf{H}), & \mathcal{S}_{\mu\nu} &= (-(\mathbf{Q} \wedge \mathbf{R}), \mathbf{Q}R_t - \mathbf{R}Q_t), \\
\mathcal{T}^{\mu\nu} &= (-\mathbf{E}, -\mathbf{H}), & \mathcal{S}^{\mu\nu} &= (-(\mathbf{Q} \wedge \mathbf{R}), -(\mathbf{Q}R_t - \mathbf{R}Q_t)). \quad (3)
\end{aligned}
$$

190. Construction of force-vector. To generate a contravariant force-vector from $(\mathbf{H}, \mathbf{E})$ or $T_{\mu\nu}$ and the contravariant 4-vector $(\mathbf{V}/Y^{\frac12}, c/Y^{\frac12})$ or say V^ν, consider the expression

$$F^\alpha = g^{\alpha\mu}V^\nu T_{\mu\nu}.$$

This is the 4-vector

$$\left(c\frac{\mathbf{E}}{Y^{\frac12}} + \frac{\mathbf{V} \wedge \mathbf{H}}{Y^{\frac12}}, \frac{\mathbf{E}.\mathbf{V}}{Y^{\frac12}}\right). \quad (4)$$

If we write simply (k being an arbitrary scalar)

$$\mathbf{F} = \frac{k}{Y^{\frac12}}\left(\mathbf{E} + \frac{\mathbf{V} \wedge \mathbf{H}}{c}\right), \qquad F_t = k\frac{\mathbf{E}.\mathbf{V}}{cY^{\frac12}}, \quad (5)$$

and introduce this value of $(\mathbf{F}, F_t)$ in the energy formulae of Chapter VI, namely

$$\mathbf{F}.\left(\frac{\mathbf{V}}{Y^{\frac12}} - \mathbf{P}\frac{Y^{\frac12}}{Z}\right) - F_t\left(\frac{c}{Y^{\frac12}} - ct\frac{Y^{\frac12}}{Z}\right) = \frac{1}{Y^{\frac12}}\frac{d}{dt}(mc^2\xi^{\frac12}), \quad (6)$$

$$F_t\frac{c}{Y^{\frac12}} - \mathbf{F}.\frac{\mathbf{V}}{Y^{\frac12}} = \frac{1}{Y^{\frac12}}\frac{d}{dt}(mc^2\xi^{\frac12}), \quad (7)$$

we obtain from the first

$$\frac{\mathbf{E}.(\mathbf{V}t - \mathbf{P})}{Z} - \frac{\mathbf{P} \wedge \mathbf{V}.\mathbf{H}}{Z} = \frac{1}{Y^{\frac12}}\frac{d}{dt}(mc^2\xi^{\frac12}), \quad (8)$$

and from the second $\qquad 0 = \dfrac{1}{Y^{\frac12}}\dfrac{d}{dt}(mc^2\xi^{\frac12}). \quad (9)$

These are inconsistent. Accordingly (5) is inadequate as a possible form of the mechanical force-vector due to $(\mathbf{H}, \mathbf{E})$.

Our previous mathematical experience of the use of integrals (6) and (7) suggests that they will become consistent if we add to the right-hand sides of (5) terms which represent the effect of the change of mass with velocity. We accordingly set tentatively

$$\mathbf{F} = \frac{k}{Y^{\frac{1}{2}}}\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right) + \alpha\frac{\mathbf{V}}{Y^{\frac{1}{2}}}, \tag{10}$$

$$F_t = \frac{k}{Y^{\frac{1}{2}}}\frac{\mathbf{E}.\mathbf{V}}{c} + \alpha\frac{c}{Y^{\frac{1}{2}}}. \tag{10'}$$

Inserting these in (6) and (7) we find that we recover (8), and (7) is then satisfied if for α we take

$$\alpha = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}). \tag{11}$$

Accordingly we put now

$$\mathbf{F} = \frac{k}{Y^{\frac{1}{2}}}\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right) + \frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{12}$$

$$F_t = \frac{k}{Y^{\frac{1}{2}}}\frac{\mathbf{E}.\mathbf{V}}{c} + \frac{c}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{12'}$$

and the energy-formula reduces to

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) = \frac{k}{Z}\left\{\mathbf{E}.(\mathbf{V}l - \mathbf{P}) - \frac{\mathbf{P}\wedge\mathbf{V}.\mathbf{H}}{c}\right\}. \tag{13}$$

This may again be written

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) = \frac{kl}{Z}\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right).\left(\mathbf{V} - \frac{\mathbf{P}}{l}\right). \tag{14}$$

To effect an identification of $\mathbf{E}$, $\mathbf{H}$ with the electric and magnetic intensities, we shall want (14) to represent the rate of performance of work by the Larmor-Lorentz ponderomotive force

$$e\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right), \tag{15}$$

in pushing the particle relative to its immediate cosmical surroundings with relative velocity $\mathbf{V} - \mathbf{P}/l$. We want therefore to remove the denominator Z without introducing any more functions involving position. Hence we put

$$k = e\frac{Z}{l_0 Y^{\frac{1}{2}}},$$

where t_0 is at present an arbitrary constant of the dimensions of a *time*. Finally then, the ponderomotive force is

$$\mathbf{F} = \frac{e}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\left(\frac{\mathbf{E}}{Y^{\frac{1}{2}}}+\frac{\mathbf{V}\wedge\mathbf{H}}{cY^{\frac{1}{2}}}\right)+\frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{10}$$

with

$$F_t = \frac{e}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\left(\frac{\mathbf{E}.\mathbf{V}}{cY^{\frac{1}{2}}}\right)+\frac{c}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{10'}$$

and the energy equation is

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) = e\frac{t}{t_0}\frac{1}{Y^{\frac{1}{2}}}\left(\mathbf{E}+\frac{\mathbf{V}\wedge\mathbf{H}}{c}\right).\left(\mathbf{V}-\frac{\mathbf{P}}{t}\right). \tag{17}$$

The multiplier e we shall later identify with the *charge* on the particle considered.

191. Motion of a pair of charges in one another's presence and the presence of the substratum. The equations of motion of a charge e_1 now reduce to

$$\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}\left(\frac{\mathbf{V}}{Y_1^{\frac{1}{2}}}\right) = -\frac{m_1\xi_1^{\frac{1}{2}}}{X_1}\left(\mathbf{P}_1-\mathbf{V}_1\frac{Z_1}{Y_1}\right)+\frac{e_1}{t_0}\frac{Z_1}{Y_1^{\frac{1}{2}}}\left(\mathbf{E}_1+\frac{\mathbf{V}_1\wedge\mathbf{H}_1}{c}\right)\frac{1}{Y_1^{\frac{1}{2}}}, \tag{18}$$

and

$$\frac{m_1\xi_1^{\frac{1}{2}}}{Y_1^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{c}{Y_1^{\frac{1}{2}}}\right) = -\frac{m_1\xi_1^{\frac{1}{2}}}{X_1}\left(ct_1-c\frac{Z}{Y_1}\right)+\frac{e_1}{t_0}\frac{Z_1}{Y_1^{\frac{1}{2}}}\left(\frac{\mathbf{E}_1.\mathbf{V}_1}{c}\right)\frac{1}{Y_1^{\frac{1}{2}}}, \tag{18'}$$

and we have a similar pair of equations with every symbol suffixed 2 instead of 1. The term in $(\mathbf{F}, F_t)$ representing the effect of change of mass with velocity has been used to cancel a term on each left-hand side. We notice here an essential difference between electromagnetic forces of the kind we have introduced here and gravitational forces, which involved a term in the change of mass with velocity of double the amount we have obtained here. We shall have later to show how to write down equations of motion containing both electrodynamic and local gravitational forces. We also note the occurrence of t_0, which made no appearance in the purely gravitational t-equations of motion.

Equation (18'), though it contains a 3-scalar term $\mathbf{E}.\mathbf{V}$, must not be mistaken for an energy equation; the actual energy equation is (17). Equation (18') follows from (18) on scalar multiplication by $\mathbf{V}_1/c$. The energy equation (17) follows also from (18) and (18') on multiplying (18) scalarly by

$$\frac{\mathbf{V}_1}{Y_1^{\frac{1}{2}}}-\mathbf{P}_1\frac{Y_1^{\frac{1}{2}}}{Z_1},$$

and (18') by

$$\frac{c}{Y_1^{\frac{1}{2}}}-ct_1\frac{Y_1^{\frac{1}{2}}}{Z_1}.$$

192. Use of super-potentials. Introduce into the energy equation (17) the values of the intensities $\mathbf{E}_1$ and $\mathbf{H}_1$ given by their expressions in terms of the super-potential ϕ_{21}. We obtain

$$\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt}(m_1 c^2\xi_1^{\frac{1}{2}}) = \frac{e_1}{t_0}\frac{1}{Y_1^{\frac{1}{2}}}\left\{\sum_{x,y,z}\frac{u_1 t_1 - x_1}{c}\left(\frac{\partial^2\phi_{21}}{\partial x_1\partial t_2} - \frac{\partial^2\phi_{21}}{\partial x_2\partial t_1}\right) - \right.$$
$$\left. - \sum_{x,y,z}\frac{y_1 w_1 - z_1 v_1}{c}\left(\frac{\partial^2\phi_{21}}{\partial y_1\partial z_2} - \frac{\partial^2\phi_{21}}{\partial z_1\partial y_2}\right)\right\},$$

where we have written (x_1, y_1, z_1) for $\mathbf{P}_1$ and (u_1, v_1, w_1) for $\mathbf{V}_1$. We rearrange this equation in the form

$$\frac{d}{dt_1}(m_1 c^2\xi_1^{\frac{1}{2}}) + \frac{e_1}{ct_0}\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + w_1\frac{\partial}{\partial z_1}\right)\times$$
$$\times\left(-t_1\frac{\partial}{\partial t_2} - x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2} - z_1\frac{\partial}{\partial z_2}\right)\phi_{21}$$
$$= -\frac{e_1}{ct_0}\left(\frac{\partial}{\partial t_2} + u_1\frac{\partial}{\partial x_2} + v_1\frac{\partial}{\partial y_2} + w_1\frac{\partial}{\partial z_2}\right)\times$$
$$\times\left\{\phi_{21} + \left(t_1\frac{\partial}{\partial t_1} + x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + z_1\frac{\partial}{\partial z_1}\right)\phi_{21}\right\}.$$

If we write
$$L_{\mu\nu} = t_\mu\frac{\partial}{\partial t_\nu} + \mathbf{P}_\mu\cdot\frac{\partial}{\partial\mathbf{P}_\nu} \quad (\mu,\nu = 1,2), \tag{19}$$

the foregoing relation may be written

$$\frac{d}{dt_1}(m_1 c^2\xi_1^{\frac{1}{2}}) + \frac{e_1}{ct_0}\left(\frac{\partial}{\partial t_1} + \mathbf{V}_1\cdot\frac{\partial}{\partial\mathbf{P}_1}\right)(-L_{12}\phi_{21})$$
$$= -\frac{e_1}{ct_0}\left(\frac{\partial}{\partial t_2} + \mathbf{V}_1\cdot\frac{\partial}{\partial\mathbf{P}_2}\right)(\phi_{21} + L_{11}\phi_{21}).$$

Writing down the corresponding relation for particle 2 by interchanging the suffixes 1 and 2, multiplying the two relations respectively by dt_1/dt, dt_2/dt, and adding, we get

$$\frac{d}{dt_1}(m_1 c^2\xi_1^{\frac{1}{2}})\frac{dt_1}{dt} + \frac{d}{dt_2}(m_2 c^2\xi_2^{\frac{1}{2}})\frac{dt_2}{dt} + \frac{e_1}{ct_0}\frac{dt_1}{dt}\left(\frac{\partial}{\partial t_1} + \mathbf{V}_1\cdot\frac{\partial}{\partial\mathbf{P}_1}\right)(-L_{12}\phi_{21}) +$$
$$+ \frac{e_2}{ct_0}\frac{dt_2}{dt}\left(\frac{\partial}{\partial t_2} + \mathbf{V}_2\cdot\frac{\partial}{\partial\mathbf{P}_2}\right)(-L_{21}\phi_{12})$$
$$= -\frac{e_1}{ct_0}\frac{dt_1}{dt}\left(\frac{\partial}{\partial t_2} + \mathbf{V}_1\cdot\frac{\partial}{\partial\mathbf{P}_2}\right)(\phi_{21} + L_{11}\phi_{21}) -$$
$$- \frac{e_2}{ct_0}\frac{dt_2}{dt}\left(\frac{\partial}{\partial t_1} + \mathbf{V}_2\cdot\frac{\partial}{\partial\mathbf{P}_1}\right)(\phi_{12} + L_{22}\phi_{12}). \tag{20}$$

Here t is a parameter which it is unnecessary at this stage to specify, a parameter giving a rule of simultaneity for events at P_1 and P_2. Some such rule is clearly necessary, as in all many-particle problems, before we can attach a meaning to *adding* together the energies of two discrete particles.

The foregoing equation should be capable of expressing the rate of change of the sum of the mechanical energies $m_1 c^2 \xi_1^{\frac{1}{2}}$ and $m_2 c^2 \xi_2^{\frac{1}{2}}$ of the two particles in terms of the new (electrical) characteristics involved in the parameters e_1 and e_2. To make the left-hand side the complete time-differential of a function of the variables involved, we now endeavour to choose the super-potentials ϕ_{21} and ϕ_{12} in such a way that

$$-\frac{e_1}{ct_0} L_{12} \phi_{21} = -\frac{e_2}{ct_0} L_{21} \phi_{12} = \Phi, \tag{21}$$

say, where Φ is symmetrical in the suffixes 1 and 2. Then, since

$$\frac{d\Phi}{dt} = \frac{dt_1}{dt}\left(\frac{\partial\Phi}{\partial t_1} + V_1 \cdot \frac{\partial\Phi}{\partial P_1} + \dot{V}_1 \cdot \frac{\partial\Phi}{\partial V_1}\right) + \frac{dt_2}{dt}\left(\frac{\partial\Phi}{\partial t_2} + V_2 \cdot \frac{\partial\Phi}{\partial P_2} + \dot{V}_2 \cdot \frac{\partial\Phi}{\partial V_2}\right),$$

our energy-relation may now be written

$$\frac{d}{dt}(m_1 c^2 \xi_1^{\frac{1}{2}} + m_2 c^2 \xi_2^{\frac{1}{2}} + \Phi) = \dot{V}_1 \cdot \frac{\partial\Phi}{\partial V_1}\frac{dt_1}{dt} + \dot{V}_2 \cdot \frac{\partial\Phi}{\partial V_2}\frac{dt_2}{dt} -$$

$$-\frac{e_1}{ct_0}\frac{dt_1}{dt}\left(\frac{\partial}{\partial t_2} + V_1 \cdot \frac{\partial}{\partial P_2}\right)(\phi_{21} + L_{11}\phi_{21}) -$$

$$-\frac{e_2}{ct_0}\frac{dt_2}{dt}\left(\frac{\partial}{\partial t_1} + V_2 \cdot \frac{\partial}{\partial P_1}\right)(\phi_{12} + L_{22}\phi_{12}). \tag{22}$$

Let us now endeavour further to choose ϕ_{21} and ϕ_{12} in such a way that the terms on the right-hand side not depending on the accelerations $\dot{V}_1$ and $\dot{V}_2$ all vanish. This requires

$$L_{11}\phi_{21} = -\phi_{21}, \qquad L_{22}\phi_{12} = -\phi_{12}. \tag{23}$$

These conditions will ensure that when the accelerations vanish, something we propose to identify as the total energy will remain constant. Conditions (23) will be satisfied if ϕ_{21} is a homogeneous function of degree -1 in the variables P_1, t_1 and ϕ_{12} a homogeneous function of degree -1 in the variables P_2, t_2.

193. Explicit forms of the super-potentials. The super-potential ϕ_{21} is to determine the field at (P_1, t_1) due to e_2 at (P_2, t_2). It will therefore be expected to have a singularity at $P_1 = P_2$, $t_1 = t_2$.

Our previous work on gravitation (Chapter X) suggests that we consider the 4-scalar

$$\frac{1}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}. \tag{24}$$

This as it stands is of degree -1 in P_1, t_1, and it will remain a possible ϕ_{21} even if multiplied by any 4-scalar not containing P_1, t_1. We have now to consider how to satisfy (21). It is clearly sufficient if $-(e_1/ct_0)L_{12}\phi_{21}$ is symmetrical in the suffixes 1, 2. By direct differentiation we find that

$$L_{12}\frac{1}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}} \equiv 0, \tag{25}$$

and hence $(X_{12}^2 - X_1 X_2)^{-\frac{1}{2}}$ behaves as a constant under the linear operator L_{12}. Hence if we put tentatively

$$\phi_{21} = e_2 \frac{\psi_{21}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}, \qquad \phi_{12} = e_1 \frac{\psi_{12}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}, \tag{26}$$

where ψ_{12}, ψ_{21} are 4-scalars, (21) will be satisfied if ψ_{21}, ψ_{12} are chosen to satisfy

$$L_{12}\psi_{21} = L_{21}\psi_{12}, \tag{27}$$

and at the same time ψ_{21} does not contain the 4-vector (P_1, ct_1) and ψ_{12} does not contain the 4-vector (P_2, ct_2).

To find a simple solution of this identity, put

$$Z_1 = t_1 - P_1 . V_1/c^2, \qquad Z_2 = t_2 - P_2 . V_2/c^2, \tag{28}$$

$$Z_{12} = t_1 - P_1 . V_2/c^2, \qquad Z_{21} = t_2 - P_2 . V_1/c^2. \tag{28'}$$

Then $Z_1/Y_1^{\frac{1}{2}}, Z_2/Y_2^{\frac{1}{2}}, Z_{12}/Y_2^{\frac{1}{2}}, Z_{21}/Y_1^{\frac{1}{2}}$ are 4-scalars, being 4-scalar products of a position-epoch vector with a velocity-vector. We notice that

$$L_{12}Z_2 = Z_{12}, \qquad L_{12}Z_{21} = Z_1,$$

$$L_{21}Z_1 = Z_{21}, \qquad L_{21}Z_{12} = Z_2.$$

Hence $\quad L_{12}(Z_2 Z_{21}) = Z_{12}Z_{21} + Z_1 Z_2 = L_{21}(Z_1 Z_{12}). \tag{29}$

(L_{12} and L_{21} are in fact interchange operators when acting on expressions linear in t_2, P_2 and t_1, P_1 respectively, replacing 2 by 1 and 1 by 2.) It follows that we may take ψ_{21} to be proportional to $Z_2 Z_{21}/Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}$, ψ_{12} proportional to $Z_1 Z_{12}/Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}$. We shall now make a special choice of a constant of proportionality with a view to later identification, and write definitively

$$\phi_{21} = -\tfrac{1}{2}e_2 \frac{Z_2 Z_{21}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}} Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}}, \tag{30}$$

$$\phi_{12} = -\tfrac{1}{2}e_1 \frac{Z_1 Z_{12}}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}} Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}}, \tag{30'}$$

whence

$$\Phi = \frac{1}{2}\frac{e_1 e_2}{ct_0}\frac{Z_{12}Z_{21}+Z_1 Z_2}{(X_{12}^2-X_1 X_2)^{\frac{1}{2}}Y_1^{\frac{1}{2}}Y_2^{\frac{1}{2}}}. \tag{31}$$

The equation we expect to be an energy equation now runs

$$\frac{d}{dt}(m_1 c^2\zeta_1^{\frac{1}{2}}+m_2 c^2\zeta_2^{\frac{1}{2}}+\Phi) = +\frac{dV_1}{dt_1}\frac{\partial\Phi}{\partial V_1}\frac{dt_1}{dt}+\frac{dV_2}{dt_2}\frac{\partial\Phi}{\partial V_2}\frac{dt_2}{dt}. \tag{32}$$

Our choice of the conditions governing ϕ_{21} and ϕ_{12} has been determined by a desire to make the right-hand side of (22) vanish when the accelerations $\dot{V}_1$ and $\dot{V}_2$ vanish. It must be recognized that we are *constructing* an electrodynamics in much the same way that a pure mathematician constructs a geometry. What we construct is partly at our disposal. Our aim is to construct something containing features which correspond to features of classical electrodynamics. With this object we have *imposed* (21) and (27) in order to force the energy equation into the form (32). This will give the rate of change of total energy as a linear function of the accelerations of the two charges provided Φ can be identified as the electrical energy associated with the two charges in one another's presence, in which case the right-hand side of (32) will give the negative of the rate of radiation of energy, $-dR/dt$. We have then

$$\frac{d}{dt}(m_1 c^2\zeta_1^{\frac{1}{2}}+m_2 c^2\zeta_2^{\frac{1}{2}}+\Phi) = -\frac{dR}{dt}, \tag{33}$$

where

$$\frac{dR}{dt} = -\dot{V}_1\cdot\frac{\partial\Phi}{\partial V_1}\frac{dt_1}{dt}-\dot{V}_2\cdot\frac{\partial\Phi}{\partial V_2}\frac{dt_2}{dt}. \tag{34}$$

194. Distinction from gravitation. Before we pass on, we remark that a very simple solution of identity (21) would have been given by

$$\psi_{21} = X_2, \qquad \psi_{12} = X_1.$$

For then we should have had

$$L_{12}X_2 = 2X_{12} = L_{21}X_1,$$

and for Φ we should have had a 4-scalar proportional to

$$\frac{X_{12}}{(X_{12}^2-X_1 X_2)^{\frac{1}{4}}}.$$

But this is just the expression we obtained in Chapter X for the mutual *gravitational* energy χ of the two particles. We should have merely recovered our gravitational theory. As a check, we notice that since

this value for Φ would not contain V_1 or V_2, the rate of radiation dR/dt would be zero, and so the sum of the mechanical energies of the particles and their gravitational energy would be conserved, as we found in Chapter X. There is thus no radiation of gravitational energy, on the present exposition of kinematic relativity. It is clear, in fact, that to get something beyond gravitation, we need a solution of (21) and (27) involving the *velocities* of the particles. This requirement is met by the solutions (30), (31).

195. Explicit forms of field-intensities. It now remains to inquire whether the values of **E** and **H** derived from ϕ_{21} by means of (1) and (2) reproduce the properties associated observationally with electric and magnetic fields. We therefore consider the 6-vector $(\mathbf{H_1}, \mathbf{E_1})$ defined by

$$(E_1)_x = \frac{1}{c}\left(\frac{\partial^2 \phi_{21}}{\partial x_1\, \partial t_2} - \frac{\partial^2 \phi_{21}}{\partial x_2\, \partial t_1}\right), \tag{35}$$

$$(H_1)_x = \left(\frac{\partial^2 \phi_{21}}{\partial y_1\, \partial z_2} - \frac{\partial^2 \phi_{21}}{\partial z_1\, \partial y_2}\right). \tag{36}$$

We wish to see whether this is the field produced at e_1 by e_2.

At first sight, the outlook is unpromising. Since $(X_{12}^2 - X_1 X_2)^{-\frac{1}{2}}$ plays the part of $1/r$, the double partial differentiation of the product of this with a scalar numerator would normally yield terms in $1/r$, $1/r^2$, $1/r^3$. But now a miracle occurs which we did not arrange for. The prospective terms in $1/r^3$ fail to appear on account of the unexpected identities

$$\left(\frac{\partial^2}{\partial y_1\, \partial z_2} - \frac{\partial^2}{\partial z_1\, \partial y_2}\right)\frac{1}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}} = 0,$$

$$\left(\frac{\partial^2}{\partial x_1\, \partial t_2} - \frac{\partial^2}{\partial x_2\, \partial t_1}\right)\frac{1}{(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}} = 0,$$

whilst terms in $1/r$ fail to appear on account of the identities

$$\left(\frac{\partial^2}{\partial y_1\, \partial z_2} - \frac{\partial^2}{\partial z_1\, \partial y_2}\right)Z_2 Z_{21} = 0,$$

$$\left(\frac{\partial^2}{\partial x_1\, \partial t_2} - \frac{\partial^2}{\partial x_2\, \partial t_1}\right)Z_2 Z_{21} = 0.$$

We are left only with terms comparable to $1/r^2$, and the inverse

square law of Coulomb appears. We find in fact from (35) and (36)

$$E_1 = \frac{e_2}{c^3 Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{(P_1 X_2 - P_2 X_{12})\frac{1}{2}(Z_2 + Z_{21}) - (t_1 X_2 - t_2 X_{12})\frac{1}{2}(V_1 Z_2 + V_2 Z_{21})}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}},$$

(37)

$$H_1 = \frac{e_2}{c^4 Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{\frac{1}{2}(Z_{21} V_2 + Z_2 V_1) \wedge (X_2 P_1 - X_{12} P_2)}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}}.$$

(38)

To see the meaning of these apparently complicated formulae, take the observer O to be at P_2, the source. Then $P_2 = 0$, $X_2 = t_2^2$, $X_{12} = t_1 t_2$, $Z_{21} = t_2$, $Z_2 = t_2$ and $(X_{12}^2 - X_1 X_2)^{-\frac{3}{2}} = c^3/t_2^3 |P_1|^3$. We find then

$$E_1 = \frac{e_2}{Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{P_1}{|P_1|^3},$$

(39)

$$H_1 = \frac{e_2}{c Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{1}{2}(V_1 + V_2) \wedge \frac{P_1}{|P_1|^3}.$$

(40)

As regards their distance-factors, these are exact inverse square law formulae. They contain in addition the velocity denominators $Y_1^{-\frac{1}{2}} Y_2^{-\frac{1}{2}}$. But (39) is sufficient to identify e_2 as the electrostatic charge at P_2.

196. Properties of the field-intensities.

Several features of these formulae are of interest. In the first place, t_2 disappears when O is taken at P_2. Thus for the observer for whom the inverse square law is exact, the only epoch that is relevant is the epoch at P_1, the test-charge. This value t_1 is the instant in O's experience to which the formulae refer, but even this has disappeared in formulae (39) and (40). We see that for an exact statement of the Coulomb inverse square law, the observer must always be explicitly mentioned, as being at the field-originating charge e_2.

As regards the velocity denominators, $Y_1^{-\frac{1}{2}} Y_2^{-\frac{1}{2}}$, we shall see the effects of these when we come to evaluate the electrodynamic forces in a simple model atom.

It must be borne in mind that (39) and (40) refer to ideal point-charges. The point-charge in this formulation has no structure—it is just a singularity. These formulae for E and H have been derived in connexion with a definite dynamics, and we shall show in due course how when the interaction of two *point*-charges is considered with the aid of this dynamics, the point-charge behaves as though it had a well-defined radius.

The formula (40) for the magnetic intensity at P_1 due to a charge e_2 at P_2 moving with velocity V_2, as measured by a test-charge e_1 at P_1 moving with velocity V_1, the velocities being measured relative to an observer at cosmical rest at P_2, is generally akin to the law of the magnetic effect of a moving charge usually attributed to Biot and Savart. It reduces in fact to the latter law when we take $V_1 = V_2$, Biot and Savart's law being of the form

$$\mathbf{H}_1 = \frac{e_2}{c} \mathbf{V}_2 \wedge \frac{\mathbf{P}_1}{|\mathbf{P}_1|^3}.$$

But in formula (40), the V_2 of Biot and Savart's law is replaced by $\frac{1}{2}(V_1 + V_2)$, V_1 being the velocity of the test-charge, V_2 the velocity of the source-charge. This means *prima facie* that the test-charge itself contributes to the magnetic field it is measuring. In other words, the magnetic field at a point P_1 is only definite when the velocity of the test-charge used to measure it is specified, and then the value attributed to the magnetic field depends on the velocity of the test-charge. This becomes plausible when we consider that the motion of the test-charge relative to the source-charge makes the test-charge appear to be in the field of a current element at the source-charge, and the magnetic field of this current element must be in evidence at P_1. In fact we may write

$$\tfrac{1}{2}(V_2 + V_1) = V_2 + \tfrac{1}{2}(V_1 - V_2), \tag{41}$$

which shows at once that over and above the Biot–Savart contribution to the magnetic field at P_1, due to the motion V_2 of the source-charge at P_2, there is a contribution depending on the excess velocity of the test-charge over that of the source-charge, of amount one-half the Biot–Savart value that would be calculated from this relative velocity. In this complicated question of the meaning of a magnetic intensity due to moving point-charges, there are three significant velocities concerned: the velocity of the observer, the velocity of the source-charge, and the velocity of the test-charge, and a proper assessment of the magnetic field depends on a correct relativistic treatment of all three velocities. We have throughout taken the observer as at local cosmical rest; and for a general position of the observer all effects will be correctly taken into effect by formulae (37) and (38). When the observer is made to coincide in position (though not in velocity) with the source-charge e_2, the electric and magnetic intensities are given by (39) and (40) wherein still two

velocities appear in the magnetic intensity, apart from the relativistic denominators $Y_1^{-1}Y_2^{-1}$. There is thus no such thing as an objective magnetic field independent of test-charge; in the absence of, or (perhaps we should say) owing to the non-existence of, discrete magnetic poles, a magnetic intensity can only be introduced as a component of the mechanical force acting on a moving test-charge, and its value then depends on the circumstances of motion of the test-charge.

The effect here found generally was first discovered in a particular case in 1926 by L. H. Thomas, in treating relativistically the motion of an electron round a proton. Relative to the electron, the proton is moving round it Ptolemaic fashion, and contributing a magnetic field at the electron; it accordingly has an influence on the motion of the electron over and above the simple Coulomb attraction. L. H. Thomas also found the factor $\frac{1}{2}$ in this particular case, as multiplying the effect that would be calculated by a crude application of the Biot–Savart formula. For $V_2 = 0$, the intensity H_1 at P_1 measured by e_1 is proportional to $\frac{1}{2}V_1$, i.e. for an atomic nucleus at rest, the effective magnetic field corresponds to one-half the linear speed of the orbital electron. We shall consider the effects of this later. I content myself with remarking at this stage that since 1926 Thomas's effect has been rather lost sight of, owing to the attribution of *spin* to electrons, with a consequent magnetic moment. In my view, the various effects supposed to be due to electron-spin are in truth consequences of the non-objective existence of magnetic fields due to moving charged particles. Just as in current spectroscopic theory each electron's spin is supposed to contribute its share to the total angular momentum of the atomic system, in my view each electron is really contributing to the magnetic field at itself by its own orbital velocity. In some way, it seems to me, the part played by electron-spin in current theory is really played by the electron's own velocity. It is in any case difficult to maintain the concept of electron-spin and at the same time treat the electron as a point-singularity, and the rational course seems to be to abandon the self-contradictory notion of electron-spin.

If the intensity of a magnetic field depends on the velocity V_1 of the test-charge used to measure it, how comes it about, the reader will ask, that we can measure the magnetic field of a permanent magnet? Now the magnetic intensity H_1 at P_1 due to a system of

moving charges e_2, e_3,... at P_2, P_3 ..., moving with velocities V_2, V_3,... as measured by e_1 moving with velocity V_1, will be approximately, apart from minor relativistic refinements,

$$H_1 = \sum_{s=2}^{n} \frac{e_s}{c} \tfrac{1}{2}(V_1 + V_s) \wedge \frac{P_1 - P_s}{|P_1 - P_s|^3}. \tag{42}$$

But the electric intensity E_1 is given by

$$E_1 = \sum_{s=2}^{n} e_s \frac{P_1 - P_s}{|P_1 - P_s|^3}. \tag{43}$$

Hence

$$H_1 = \frac{1}{2} \sum_{s=2}^{n} \frac{e_s}{c} V_s \wedge \frac{P_1 - P_s}{|P_1 - P_s|^3} + \frac{1}{2} \frac{V_1 \wedge E_1}{c}. \tag{44}$$

If therefore the system of moving charges, e_2,..., e_n responsible for the magnetic field at P_1 is electrostatically neutral as regards its resultant electrostatic effect at P_1, the second term on the right-hand side of the above equation vanishes ($E_1 = 0$), and H_1 is then independent of V_1. There is therefore in this case an objective magnetic field. This accounts for the existence of a magnetic field from such a body as a permanent magnet, which is, of course, electrostatically neutral in its effects at points external to itself. The strength of the resulting magnetic field H_1 is then just one-half what would be calculated by the Biot-Savart law from the charges in motion originating the magnetic field. It is clear that only in the interiors of atomic systems or systems of ions can we expect to find the discrepancies with classical electromagnetic theory predicted by the present theory. An example of this is the gyro-magnetic effect, where the factor $\tfrac{1}{2}$ turns up experimentally. Again, in Stoner's *Magnetism* (1930) it is suggested that the magnetization of ferro-magnetics is entirely due to 'intrinsic spin' of the electrons, and that their orbital moment is not effective; the present investigation suggests that the explanation of the effect is that there is no such thing as electron-spin, and that the effect arises purely from the orbital motions of the electrons.

197. Mechanical forces due to electric and magnetic intensities.

By formula (10) the contribution to the mechanical force F due to the electric intensity E at e is

$$F = \frac{e}{l_0} \frac{Z}{Y^{\frac{1}{2}}} \frac{E}{Y^{\frac{1}{2}}}. \tag{45}$$

For $|P_1| \ll ct_1$, $|V_1| \ll c$, this is approximately

$$F \sim e \frac{t}{t_0} E.$$ (46)

Hence the contribution to F due to a single charge e_2 at rest at the origin is given by

$$F_1 \sim e_1 e_2 \frac{t_1}{t_0} \frac{P_1}{|P_1|^3}.$$ (47)

This is the measure of F_1 in t-measure, in which we have worked exclusively so far in this chapter. Now the τ-measure Φ of the same force we saw in § 96 to be given by

$$\Phi_1 \sim \frac{t_1}{t_0} F_1.$$ (48)

Hence

$$\Phi_1 \sim \left(\frac{t_1}{t_0}\right)^2 e_1 e_2 \frac{P_1}{|P_1|^3}.$$

But

$$P_1 = \frac{t_1}{t_0} \Pi_1,$$

where Π_1 is the τ-measure of P_1. Hence

$$\Phi_1 \sim e_1 e_2 \frac{\Pi_1}{|\Pi_1|^3}.$$ (49)

Thus in τ-measure the secular factor t_1/t_0 disappears, and we get the empirical Coulomb inverse square law.

The contribution to the mechanical force due to the magnetic intensity is

$$F_1 \sim e_1 \frac{t_1}{t_0} \frac{V_1 \wedge H_1}{c},$$

which when the observer is at P_2 amounts to

$$F_1 \sim \frac{e_1 e_2}{c^2} \frac{t_1}{t_0} V_1 \wedge \frac{\frac{1}{2}(V_1 + V_2) \wedge P_1}{|P_1|^3}.$$ (50)

The secular factor t_1/t_0 will again disappear when we use τ-measure.

198. Electromagnetic energy. The term we have called Φ was evaluated in formula (31). For $|P_1| \ll ct_1$, $|P_2| \ll ct_2$, in virtue of the identity

$$(X_{12}^2 - X_1 X_2) = \frac{1}{c^2} \left\{ (t_1 P_2 - t_2 P_1)^2 - \frac{(P_1 \wedge P_2)^2}{c^2} \right\},$$ (51)

(31) reduces approximately to

$$\Phi \sim e_1 e_2 \frac{t_1 t_2}{t_0} \frac{1}{|t_2 P_2 - t_2 P_1| Y_1 Y_2}.$$ (52)

This should represent the electromagnetic energy associated with the pair of charges e_1, e_2. It requires a simultaneity convention relating t_1 and t_2 before it becomes numerically definite. When, however, we take the origin or observer to be at P_2, so that $\mathbf{P}_2 = 0$, it reduces exactly to

$$\Phi = e_1 e_2 \frac{t_1}{t_0} \frac{1}{|\mathbf{P}_1|Y_1^{\frac{1}{2}}Y_2^{\frac{1}{2}}|}, \tag{53}$$

which in turn reduces to the Coulomb electrostatic energy when the velocities are zero and t_1 reduces to t_0. But it is to be noted that when $\mathbf{V}_1 = 0$ and $\mathbf{V}_2 = 0$, the charged particles, though at rest relative to the observer P_2, are not at local cosmical rest. It follows that in electrostatics, 'rest' is to be taken to mean 'rest relative to the observer', not 'local rest'.

199. Secular variation of measure of charge. It will be seen that in t-measure, Φ contains a secular factor, which for an observer at P_2 reduces to t_1/t_0. But energy as such is a secular invariant; in fact when we transform (53) to τ-measure, the value of Φ is independent of the secular factor, becoming

$$\Phi \sim \frac{e_1 e_2}{|\mathbf{\Pi}_1|}.$$

In order that Φ shall be independent of choice of t_0, we must have $e_1 e_2 \propto t_0$ in t-measure. Thus, in t-measure, the value attributed to a charge must vary as the square root of the normalization constant t_0. It is in fact a characteristic difference between gravitation and electrodynamics that the t-equations themselves involve mention of t_0. This is because we use the same number e to denote a charge in t-measure and τ-measure. In order therefore that Φ shall be independent of choice of t_0, the value attributed to a charge e must vary as $t_0^{\frac{1}{2}}$. The same feature is also seen in connexion with the fine-structure constant

$$\frac{2\pi e^2}{hc}.$$

We have seen in discussing light, Chapter VIII, that h must vary secularly with the time in t-measure, and that h_0 is proportional to t_0. This is also necessitated by the fact that angular momentum varies as t in t-measure, and so as t_0 in τ-measure. Since also $c^2 \propto t_0$, the fine-structure constant is independent both of t and of t_0, as it should be, being a pure number.

200. Dependence of magnetic intensity on velocity of test-charge relative to observer. It may be noted in conclusion of this chapter that surprising as our results concerning magnetic fields may sound, there seems no escape from making the magnetic field H_1 at P_1 depend on the velocity V_1 of the test-particle at P_1, as well as on the source velocity V_2. The 6-vector (H_1, E_1), as given by (37) and (38), is of the form $S^{\mu\nu}$ of formulae (3), with

$$Q = -\frac{Z_{21} V_2 + Z_2 V_1}{Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}},$$

$$R = \frac{X_2 P_1 - X_{12} P_2}{(X_{12}^2 - X_1 X_2)^{\frac{3}{2}}},$$

and it is difficult to see how all the requirements of relativity could be met if V_1 were absent. We have developed our treatment of electrodynamics, not by beginning with experimental laws and then adjusting them to be relativistic, but by beginning with abstract possibilities and formulating them in strictly relativistic language all through. But the results are not only strictly relativistic; they also conform to the requirements of common sense. It is impossible to use a moving test-charge to measure a magnetic field without its own velocity, relative to the velocities of other charges present in the field, bringing into apparent existence distant additional currents. Our task in the next chapter will therefore be to see how much of Maxwell's field theory survives when the magnetic field depends on the velocity of the test-charge measuring it.

XIV

FIELD THEORY

201. System of n point-charges. We have so far mainly confined our attention to a pair of point-charges in one another's presence. We now wish to consider the field produced by any finite number of point-charges. We shall call the point-charges $e_1, e_2, ..., e_n$, and regard e_1 as the test-charge measuring the field at P_1; e_1 is supposed to be moving with velocity V_1, e_2 with V_2, etc., relative to an observer O anchored to some fundamental particle O of the substratum. We consider the charge e_s to be at P_s at epoch t_s all as measured by O.

We now put

$$(E_1)_x = \sum_{s=2}^{n} \frac{1}{c}\left(\frac{\partial^2 \phi_{s1}}{\partial x_1 \partial t_s} - \frac{\partial^2 \phi_{s1}}{\partial x_s \partial t_1}\right), \tag{1}$$

$$(H_1)_x = \sum_{s=2}^{n} \left(\frac{\partial^2 \phi_{s1}}{\partial y_1 \partial z_s} - \frac{\partial^2 \phi_{s1}}{\partial z_1 \partial y_s}\right), \tag{1'}$$

where

$$\phi_{s1} = -\frac{1}{2}\frac{e_s}{Y_1^{\frac{1}{2}} Y_s^{\frac{1}{2}}}\frac{Z_s Z_{s1}}{(X_{1s}^2 - X_1 X_s)^{\frac{1}{2}}}, \tag{2}$$

with

$$X_1 = t_1^2 - P_1^2/c^2, \qquad X_s = t_s^2 - P_s^2/c^2, \qquad X_{1s} = t_1 t_s - P_1 . P_s/c^2, \tag{3}$$

$$Y_1 = 1 - V_1^2/c^2, \qquad Y_s = 1 - V_s^2/c^2, \tag{4}$$

$$Z_s = t_s - P_s . V_s/c^2, \qquad Z_{s1} = t_s - P_s . V_1/c^2. \tag{5}$$

The mechanical force acting on the test-charge e_1 is given by

$$F_1 = \frac{e_1}{t_0}\frac{Z_1}{Y_1^{\frac{1}{2}}}\left(E_1 + \frac{V_1 \wedge H_1}{c}\right)\frac{1}{Y_1^{\frac{1}{2}}} + \frac{V_1}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1 \xi_1^{\frac{1}{2}}), \tag{6}$$

$$(F_l)_1 = \frac{e}{t_0}\frac{Z_1}{Y_1^{\frac{1}{2}}}\left(\frac{E_1 . V_1}{c}\right)\frac{1}{Y_1^{\frac{1}{2}}} + \frac{c}{Y_1^{\frac{1}{2}}}\frac{1}{Y_1^{\frac{1}{2}}}\frac{d}{dt_1}(m_1 \xi_1^{\frac{1}{2}}), \tag{6'}$$

in the sense that these values for F_1 and $(F_l)_1$ introduced into the l-equations of motion, give the motion of the particle m_1. Forming the associated energy equations and reducing them as in the previous chapter, we arrive at the relation

$$\frac{d}{dt}\left(\sum_{s=1}^{n} m_s c^2 \xi_s^{\frac{1}{2}} + \Phi\right) = \sum_{s=1}^{n}\frac{dV_s}{dt_s}\frac{\partial\Phi}{\partial V_s}\frac{dt_s}{dt}, \tag{7}$$

with

$$\Phi = \sum_{r,s} \Phi_{r,s} \quad (r \neq s), \tag{8}$$

and

$$\Phi_{r,s} = \frac{1}{2}\frac{e_r e_s}{ct_0}\frac{Z_{sr} Z_{rs} + Z_r Z_s}{(X_{rs}^2 - X_r X_s)^{\frac{1}{2}} Y_r^{\frac{1}{2}} Y_s^{\frac{1}{2}}}. \tag{9}$$

It will be seen that we build up the total field by superposition of the super-potentials of each pair of charges, and that the test-charge, with its circumstances of motion, is an intrinsic member of the system. The theory has nothing to say about the values of E, H at places where there is no test-charge. The scalar Φ is to be interpreted as electromagnetic energy. There is no such thing in our presentation as the self-energy of a point-charge. Energy only appears as the rate of performance of work against the electromagnetic forces concerned.

202. Maxwell's equations. In Maxwell's theory, the equations taken as a basis, valid in free space, are

$$\operatorname{div} E = 0, \qquad \operatorname{div} H = 0, \qquad (10), (11)$$

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad \operatorname{curl} H = +\frac{1}{c}\frac{\partial E}{\partial t}. \qquad (12), (13)$$

In our present theory, we do not assume these formulae. Instead, we proceed to investigate how far these or similar identities are satisfied. We therefore look for identities satisfied by E_1, H_1 in virtue of their being derived from super-potentials.

203. Definitions of differential operators. We define the operators *div*, *curl*, by

$$\operatorname{div} H_1 \equiv (\operatorname{div})_1 H_1 = \frac{\partial (H_1)_x}{\partial x_1} + \frac{\partial (H_1)_y}{\partial y_1} + \frac{\partial (H_1)_z}{\partial z_1},$$

$$(\operatorname{curl} H_1)_x \equiv (\operatorname{curl})_1 H_1 = \frac{\partial (H_1)_z}{\partial y_1} - \frac{\partial (H_1)_y}{\partial z_1}.$$

Similarly $\qquad \operatorname{grad}\phi = \left(\dfrac{\partial \phi}{\partial x_1}, \dfrac{\partial \phi}{\partial y_1}, \dfrac{\partial \phi}{\partial z_1}\right) \equiv (\operatorname{grad})_1 \phi.$

204. Satisfaction of two of Maxwell's equations. Using (1') we have

$$\operatorname{div} H_1 = \sum_{x,y,z}\sum_{s=2}^{n}\frac{\partial}{\partial x_1}\left(\frac{\partial^2\phi_{s1}}{\partial y_1\,\partial z_s} - \frac{\partial^2\phi_{s1}}{\partial z_1\,\partial y_s}\right) \equiv 0, \qquad (14)$$

and using (1), we have

$$(\operatorname{curl} E_1)_x = \frac{1}{c}\frac{\partial}{\partial y_1}\left\{\sum_{s=2}^{n}\left(\frac{\partial^2\phi_{s1}}{\partial z_1\,\partial t_s} - \frac{\partial^2\phi_{s1}}{\partial z_s\,\partial t_1}\right)\right\} - \frac{1}{c}\frac{\partial}{\partial z_1}\left\{\sum_{s=2}^{n}\left(\frac{\partial^2\phi_{s1}}{\partial y_1\,\partial t_s} - \frac{\partial^2\phi_{s1}}{\partial y_s\,\partial t_1}\right)\right\}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t_1}\left\{\sum_{s=2}^{n}\left(\frac{\partial^2\phi_{s1}}{\partial y_1\,\partial z_s} - \frac{\partial^2\phi_{s1}}{\partial z_1\,\partial y_s}\right)\right\} = -\frac{1}{c}\frac{\partial (H_1)_x}{\partial t_1}. \qquad (15)$$

In these partial differentiations we have kept V_1, the velocity of e_1, as well as all the other velocities V_s, constant. It thus appears that two of Maxwell's equations, namely (11) and (12) above, are satisfied identically in virtue of the structure of E, H in terms of the ϕ_{s1}'s, without its being necessary to use the actual value of the ϕ_{s1}'s. We can now if we like drop the suffix 1 from (14) and (15), and use the results generally. They contain no mention of the field-charges $e_2,\ldots,e_n$. It will be recalled that Maxwell's equation (11) expresses the non-existence of magnetic charges, and (12) expresses Faraday's law of electromagnetic induction.

205. Remaining pair of Maxwell's equations. We proceed to consider Maxwell's other two equations. We have by (1)

$$\operatorname{div} \mathbf{E}_1 = \frac{1}{c} \sum_{x,y,z} \sum_{s=2}^{n} \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi_{s1}}{\partial x_1 \partial t_s} - \frac{\partial^2 \phi_{s1}}{\partial x_s \partial t_1}\right)$$

$$= \frac{1}{c} \sum_{s=2}^{n} \frac{\partial}{\partial t_s} \Box_1^2 \phi_{s1} - \frac{1}{c}\frac{\partial}{\partial t_1}\left(\sum_{s=2}^{n} \Box_1 \Box_s \phi_{s1}\right), \qquad (16)$$

where we have written $\Box_1^2$ for the Dalembertian

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t_1^2},$$

and $\Box_1 \Box_s$ for the 'mixed' Dalembertian

$$\frac{\partial^2}{\partial x_1 \partial x_s} + \frac{\partial^2}{\partial y_1 \partial y_s} + \frac{\partial^2}{\partial z_1 \partial z_s} - \frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_s}.$$

Again,

$$(\operatorname{curl} \mathbf{H}_1)_x = \frac{\partial}{\partial y_1}\left\{\sum_{s=2}^{n}\left(\frac{\partial^2 \phi_{s1}}{\partial x_1 \partial y_s} - \frac{\partial^2 \phi_{s1}}{\partial y_1 \partial x_s}\right)\right\} - \frac{\partial}{\partial z_1}\left\{\sum_{s=2}^{n}\left(\frac{\partial^2 \phi_{s1}}{\partial z_1 \partial x_s} - \frac{\partial^2 \phi_{s1}}{\partial x_1 \partial z_s}\right)\right\}$$

$$= -\sum_{s=2}^{n} \frac{\partial}{\partial x_s} \Box_1^2 \phi_{s1} + \frac{\partial}{\partial x_1}\left(\sum_{s=2}^{n} \Box_1 \Box_s \phi_{s1}\right) +$$

$$+ \frac{1}{c}\frac{\partial}{\partial t_1}\left\{\sum_{s=2}^{n} \frac{1}{c}\left(\frac{\partial^2 \phi_{s1}}{\partial x_1 \partial t_s} - \frac{\partial^2 \phi_{s1}}{\partial x_s \partial t_1}\right)\right\}.$$

Hence

$$\left(\operatorname{curl} \mathbf{H}_1 - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t_1}\right)_x = -\sum_{s=2}^{n} \frac{\partial}{\partial x_s} \Box_1^2 \phi_{s1} + \frac{\partial}{\partial x_1}\left(\sum_{s=2}^{n} \Box_1 \Box_s \phi_{s1}\right). \quad (17)$$

The further reduction of the identities (16) and (17) thus depends

on the structure of the ϕ_{s1}'s. From our work on the gravitational potential χ, we have seen that

$$\square_1^2 \frac{X_{s1}}{(X_{s1}^2 - X_s \, X_1)^{\frac{1}{2}}} \equiv 0.$$

We now find, as an identity we did not arrange for, that

$$\square_1^2 \frac{1}{(X_{s1}^2 - X_s \, X_1)^{\frac{1}{2}}} \equiv 0.$$

(The number of space-dimensions, 3, plays an essential part in the establishment of this identity.) It now follows that

$$\square_1^2 \phi_{s1} \equiv 0. \tag{17'}$$

Thus the first term on the right-hand sides of each of (16) and (17) vanishes. But the second term on each right-hand side does *not* vanish, $\square_1 \square_s \phi_{s1} \not\equiv 0$. We find in fact that

$$\square_1 \square_s \phi_{s1} = -\frac{1}{2} \frac{e_s}{Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{1}{c^2} \frac{2X_{s1} Z_s Z_{s1} - X_s(Z_s Z_1 + Z_{1s} Z_{s1})}{(X_{s1}^2 - X_s \, X_1)^{\frac{5}{2}}}.$$

That this does not vanish identically is most easily seen by taking the origin at P_s, when it reduces to

$$-\frac{1}{2} \frac{e_s}{Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}} \frac{1}{c} \frac{\mathbf{P}_1 \cdot (\mathbf{V}_1 + \mathbf{V}_s)}{|\mathbf{P}_1|^3},$$

which is of the order of magnitude of $\mathbf{H}_1$. We shall put

$$\sum_{s=2}^{n} \square_1 \square_s \phi_{s1} = a_1,$$

where a_1 is a 4-scalar. It is symmetrical in the suffixes $2, ..., n$. We have then, by (16) and (17)

$$\operatorname{div} \mathbf{E}_1 = -\frac{1}{c} \frac{\partial a_1}{\partial t_1}, \tag{18}$$

$$\operatorname{curl} \mathbf{H}_1 - \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t_1} = \operatorname{grad} a_1. \tag{19}$$

Hence
$$\left(\operatorname{curl} \mathbf{H}_1 - \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t_1}, \operatorname{div} \mathbf{E}_1\right)$$

is a contravariant 4-vector. Eliminating (a_1) between (18) and (19) by cross-differentiation, we have

$$\operatorname{grad} \operatorname{div} \mathbf{E}_1 + \frac{\partial}{c \, \partial t_1}\left(\operatorname{curl} \mathbf{H}_1 - \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t_1}\right) = 0, \tag{20}$$

$$\operatorname{curl}\left(\operatorname{curl} \mathbf{H}_1 - \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t_1}\right) = 0. \tag{21}$$

206. Summary of field-identities. Omitting the suffix 1 from the various symbols as no longer necessary, we have finally the four identities

$$\operatorname{div} \mathbf{E} = -\frac{1}{c}\frac{\partial a}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0, \qquad (22), (23)$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{curl} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{grad} a, \quad (24), (25)$$

with accordingly

$$\operatorname{grad} \operatorname{div} \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\left(\operatorname{curl} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = 0, \qquad (26)$$

$$\operatorname{curl}\left(\operatorname{curl} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = 0. \qquad (27)$$

We have already noted that (23) and (24) are identical with two of Maxwell's equations, namely (11) and (12); and that these two have simple physical meanings, namely the non-existence of magnetic charges and the law of electromagnetic induction as found by Faraday. The other two of Maxwell's equations for free space, namely (10) and (13), are on a totally different footing, as pointed out by Lorentz in 1916.† Equation (13) is a pure hypothesis, the hypothesis of the displacement current, and (10) as it stands encounters difficulties, since it would appear to proclaim the non-existence of electric charges, as (11) does of magnetic poles. It is therefore not surprising that these two relations appear modified in our treatment.

207. Comments on the field-intensities. It must be remembered that **H**, and to a lesser extent **E**, depend on the velocity of the test-charge used to measure them, in our treatment, whereas in the classical Maxwell theory **H** and **E** are conceptual quantities existing everywhere and independent of the velocity of the test-charge. In our treatment, **H** and **E** nowhere have a meaning until the test-charge is introduced *and its velocity specified*: $\mathbf{H}_1$ and $\mathbf{E}_1$ are functions of $\mathbf{V}_1$. But the identities (26) and (27) contain no mention of the velocity of the test-charge, and hold good whatever the value of $\mathbf{V}_1$, provided this is treated as constant in carrying out the partial differentiations. It should be noted also that Maxwell's equations (10) and (13) imply (26) and (27), but not conversely.

† *Theory of Electrons*, pp. 12, 239.

208. Epistemological considerations. Another difference from the classical theory is that in the classical theory the properties of the field are derived from (10)–(13), those field equations being *posited*, whilst our relations (22)–(25) and (26), (27) are end-products. In the classical theory, to determine the motions of the charges, an equation of mechanical force (obtained by using the Larmor-Lorentz ponderomotive force formula) is superposed on the set of field equations; but in our treatment we begin with equations of mechanical force, and so *derive* the forms of the super-potentials ϕ_{s1} and ϕ_{1s} from very general considerations. Our treatment seems epistemologically preferable, since to be both logically and epistemologically satisfactory the analysis should begin with the mechanical effects through which alone E and H can be known. Since on the electrical theory of matter no entity exists corresponding to an isolated magnetic pole, it is unsatisfactory to define H through the force on an isolated pole, as the classical theory does, for that is to introduce elements strange to the class of entities under consideration, namely a set of point charges. Such a procedure is not only unacceptable in our theory, it is logically impossible. Instead, we have obtained H as a constituent of the force acting on a moving point-charge.

209. Maxwell's equations and *t*-measure. Our field identities (22), (23), (26), (27) hold good in *t*-time and *t*-measure, and have been derived from an analysis of the dynamics of moving charges expressed in *t*-measure. Our analysis pays in fact due respect to the circumstance that all phenomena take place on the stage of the expanding universe, which possesses everywhere a local standard of rest; and our analysis respects Mach's principle, according to which all frames of reference employed must be described with regard to the actual distribution of matter constituting the universe. The classical theory, in ignoring the expansion of the universe, can only be supposed to hold good locally; and as it ignores the distinction between *t*- and τ-measure, it will only be likely to hold good for epochs close to the present epoch. There would in fact be grave difficulties in attempting to extend classical electrodynamics, *as it stands*, to epochs other than the present, or to distances comparable with the radius of the universe, for the field equations of classical electrodynamics being Lorentz-invariant, are stated in *t*-measure, whilst the Larmor-Lorentz ponderomotive force formula is used in contexts which employ

τ-measure. In the present treatment we have systematically used t-measure throughout, including the mechanical equations; it is only thus that we are able to recognize that Maxwell's equations, in the modified form we have obtained them, employ t-measure.

210. Wave-propagation. The evidence for the correctness of Maxwell's equations is partly direct—Faraday's demonstration of electro-magnetic induction—and partly indirect, namely the fact that the equations yield wave-propagation for the vectors E and H. We must therefore inquire next how far our field identities imply wave-propagation.

Expanding the left-hand side of (27) we get

$$\operatorname{grad} \operatorname{div} \mathbf{H} - \nabla^2 \mathbf{H} - \frac{1}{c} \frac{\partial}{\partial t}(\operatorname{curl} \mathbf{E}) = 0.$$

Using (23) and (24), this reduces to

$$-\nabla^2 \mathbf{H} + \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0,$$

or
$$\square^2 \mathbf{H} = 0. \tag{28}$$

Thus H obeys the wave-equation. Again, from (26),

$$\operatorname{curl} \operatorname{curl} \mathbf{E} + \nabla^2 \mathbf{E} + \operatorname{curl}\left(\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}\right) - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

or
$$\operatorname{curl}\left(\operatorname{curl} \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}\right) + \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

or, again using (24),
$$\square^2 \mathbf{E} = 0. \tag{29}$$

Thus E also obeys the wave-equation. In fact, the four identities replacing Maxwell's equations may be taken to be (23), (24), (28), and (29). Further, the scalar a obeys the wave-equation. For restoring temporarily the suffix 1 referring to the test-charge,

$$\square_1^2 a_1 = \square_1^2 \sum_{s=2}^{n} \square_1 . \square_s \, \phi_{s1} = \sum_{s=2}^{n} \square_1 . \square_s (\square_1^2 \phi_{s1}) \equiv 0, \tag{30}$$

by (17').

211. Vector potential. The field vectors H, E can be expressed in terms of a vector potential as follows. Define a contravariant† vector $(\mathbf{A}_1, A_1)$ by

$$\mathbf{A}_1 = \sum_{s=2}^{n} \frac{\partial \phi_{s1}}{\partial \mathbf{P}_s}, \qquad A_1 = -\sum_{s=2}^{n} \frac{\partial \phi_{s1}}{c \, \partial t_s}. \tag{31, 31'}$$

† The treatment and notation are here slightly different from those in *Proc. Roy. Soc.* **165 A**, 340, 1938.

Then by (1) and (1'),

$$E_1 = -\frac{\partial A_1}{\partial P_1} - \frac{\partial \mathbf{A}_1}{c \, \partial t_1} = -\operatorname{grad} A_1 - \frac{\partial \mathbf{A}_1}{c \, \partial t_1}, \tag{32}$$

and

$$(H_1)_x = \frac{\partial}{\partial y_1}(\mathbf{A}_1)_z - \frac{\partial}{\partial z_1}(\mathbf{A}_1)_y,$$

or

$$\mathbf{H}_1 = \operatorname{curl} \mathbf{A}_1. \tag{32'}$$

Also

$$\operatorname{div} \mathbf{A}_1 + \frac{1}{c}\frac{\partial A_1}{\partial t_1} = \sum_{s=2}^{n} \square_1 . \square_s \phi_{s1} = a_1. \tag{33}$$

Again,

$$\square_1^2 \mathbf{A}_1 = \sum_{s=2}^{n} \frac{\partial}{\partial P_s} \square_1^2 \phi_{s1} \equiv 0, \tag{34}$$

$$\square_1^2 A_1 = -\sum_{s=2}^{n} \frac{\partial}{c \, \partial t_s} \square_1^2 \phi_{s1} \equiv 0. \tag{34'}$$

We can now suppress the unnecessary suffix 1 and write as vector potential $(\mathbf{A}, A_t)$ with the properties that

$$\mathbf{E} = -\operatorname{grad} A_t - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{H} = \operatorname{curl} \mathbf{A}, \qquad (35), (35')$$

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial A_t}{\partial t} = a, \tag{36}$$

$$\square^2 \mathbf{A} = 0, \qquad \square^2 A_t = 0. \qquad (37), (37')$$

This vector potential, the 4-vector $(\mathbf{A}, A_t)$, is a kind of intermediary between the field vectors and the super-potentials. Relations (35), (35'), (37), (37') also occur in the classical theory, but in the classical theory the right hand of (36) is arbitrarily put equal to zero.

It is clear that A_t or A_1 should be equal to the electrostatic potential when the field is stationary in time. The electrostatic potential at e_1 is the energy of interaction per unit of e_1, with the other charges $e_2,...,e_n$ in the field. Hence A_1 should stand in a simple relation to $\sum_{s=2}^{n} \Phi_{s,1}$. Now by (21), of the preceding chapter,

$$\Phi_{s,1} = -\frac{e_1}{ct_0} L_{1s} \phi_{s1}, \tag{38}$$

where L_{1s} is the operator

$$P_1 . \frac{\partial}{\partial P_s} + t_1 \frac{\partial}{\partial t_s}.$$

Now take the origin at e_1, the test-charge, so that $P_1 = 0$. Then

$$\sum_{s=2}^{n} \Phi_{s1} = - \sum_{s=2}^{n} \frac{e_1 t_1}{c t_0} \frac{\partial \phi_{s1}}{\partial t_s} = e_1 \frac{t_1}{t_0} A_1. \tag{39}$$

Thus at $t_1 = t_0$, A_t is the energy associated with the charge at P_1, per unit of charge.

212. Relations between epochs. So far we have not found it necessary to state any form of correlation between the various epochs $t_2,...,t_s$ and the epoch t_1 at the test-particle. But the fact that E and H satisfy the wave-equation suggests that t_s should be taken to be the retarded position corresponding to the epoch t_1 at the test-particle. This suggests taking

$$t_1 = t_s + \frac{|P_1 - P_s|}{c}. \tag{40}$$

This gives
$$t_1 P_s - t_s P_1 = t_1(P_s - P_1) + \frac{P_1 |P_1 - P_s|}{c}. \tag{41}$$

Hence since

$$X_{1s}^2 - X_1 X_s = \frac{(t_s P_1 - t_1 P_s)^2}{c^2} - \frac{(P_1 \wedge P_s)^2}{c^4},$$

we find on eliminating t_s that

$$(X_{1s}^2 - X_1 X_s)^{\frac{1}{2}} = \frac{t_1 |P_1 - P_s|}{c} + \frac{P_1 \cdot (P_s - P_1)}{c^2}, \tag{42}$$

the surd disappearing. When we choose the origin at the test-particle, $P_1 = 0$ and we get simply

$$(X_{1s}^2 - X_1 X_s)^{\frac{1}{2}} = t_1 |P_s| / c. \tag{43}$$

Here P_s denotes the *retarded* position of the charge e_s. This will be found to give, when the origin is taken at P_1, and $|V_1| \ll c$, $|V_2| \ll c$

$$E_1 = - \sum_{s=2}^{n} e_s \frac{t_s^2}{t_1^2} \frac{P_s}{|P_s|^3}, \tag{44}$$

thus giving the Coulomb inverse square law in terms of the retarded position of P_s.

213. Combined gravitational and electromagnetic fields. When a pure gravitational field is present, we saw that the expression for the external force F on a particle of mass m contained a term

$$2 \frac{V}{Y^{\frac{1}{2}}} \frac{1}{Y^{\frac{1}{2}}} \frac{d}{dt}(m \xi^{\frac{1}{2}}),$$

in addition to the gradient of the potential. When a pure electro-
magnetic field is present, we saw that the external force F on the
particle contained a term

$$\frac{V}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}),$$

in addition to the term representing the Larmor-Lorentz pondero-
motive force. As this additional term is in the gravitational case
twice what it is in the electromagnetic case, the question arises as
to what is its value when both fields are present together.

Let χ as usual be the gravitational potential due to the point-
masses in the field. Let E, H be the electric and magnetic intensities.
Then we seek to represent the external force F on a particle of mass
m and charge e by an expression of the form

$$\mathbf{F} = -\frac{\partial\chi}{\partial\mathbf{P}} + \frac{Z}{Y^{\frac{1}{2}}}\frac{e}{t_0}\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right)\frac{1}{Y^{\frac{1}{2}}} + \alpha\frac{\mathbf{V}}{Y^{\frac{1}{2}}}, \tag{45}$$

with

$$F_t = +\frac{\partial\chi}{c\,\partial t} + \frac{Z}{Y^{\frac{1}{2}}}\frac{e}{t_0}\left(\frac{\mathbf{E}.\mathbf{V}}{c}\right)\frac{1}{Y^{\frac{1}{2}}} + \alpha\frac{c}{Y^{\frac{1}{2}}}, \tag{45'}$$

where α is a 4-scalar to be determined.

The usual energy relation

$$F_t\frac{c}{Y^{\frac{1}{2}}} - \mathbf{F}.\frac{\mathbf{V}}{Y^{\frac{1}{2}}} = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}), \tag{46}$$

with the values (45) and (45') for F and F_t, yields

$$\frac{1}{Y^{\frac{1}{2}}}\frac{d\chi}{dt} + \alpha c^2 = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}). \tag{47}$$

The other energy equation

$$\mathbf{F}.\left(\frac{\mathbf{V}}{Y^{\frac{1}{2}}} - \mathbf{P}\frac{Y^{\frac{1}{2}}}{Z}\right) - F_t\left(\frac{c}{Y^{\frac{1}{2}}} - ct\frac{Y^{\frac{1}{2}}}{Z}\right) = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}), \tag{48}$$

when we use the fact that χ is homogeneous and of degree zero in
P and ct, yields

$$\frac{e}{t_0}\left\{\mathbf{E}.(\mathbf{V}t - \mathbf{P}) - \frac{\mathbf{P}\wedge\mathbf{V}.\mathbf{H}}{c}\right\}\frac{1}{Y^{\frac{1}{2}}} = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(mc^2\xi^{\frac{1}{2}} + \chi). \tag{49}$$

Take now the particular case of a pure gravitational field. Then
E = 0, H = 0, and (47) and (49) give

$$\alpha = \frac{2}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \qquad \frac{d\chi}{dt} = -\frac{d(mc^2\xi^{\frac{1}{2}})}{dt}, \tag{50}$$

as stated above. Take secondly the case of a pure electromagnetic field (apart from the field of the substratum, which is always present), so that $\chi = 0$. Then we get

$$\alpha = \frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{51}$$

and

$$\frac{et}{t_0}\left(\mathbf{E} + \frac{\mathbf{V}\wedge\mathbf{H}}{c}\right)\cdot\left(\mathbf{V} - \frac{\mathbf{P}}{t}\right) = \frac{d}{dt}(mc^2\xi^{\frac{1}{2}}). \tag{51'}$$

Relation (50) shows that the sum of the mechanical and gravitational energies remains constant during the motion; (51') shows that the work done by the Larmor-Lorentz force in pushing the particle relative to its immediate cosmic environment reappears as mechanical energy.

When the gravitational and electromagnetic fields are associated with a set of massive charged particles, we have to use a separate epoch-coordinate t_s for each particle (m_s, e_s) and a separate coefficient α_s. The gravitational potential $\chi = \sum_{r,s} \chi_{rs}$. Forming from the equation of motion for each particle the analogue of (46), we find

$$\frac{1}{Y_s^{\frac{1}{2}}}\left(\frac{\partial\chi}{\partial t_s} + \mathbf{V}_s\cdot\frac{\partial\chi}{\partial\mathbf{P}_s}\right) + \alpha_s c^2 = \frac{1}{Y_s^{\frac{1}{2}}}\frac{d}{dt_s}(m_s c^2\xi_s^{\frac{1}{2}}),$$

and forming the analogue of (48) we find

$$\frac{e_s}{t_0}\left\{\mathbf{E}_s\cdot(\mathbf{V}_s t_s - \mathbf{P}_s) - \frac{\mathbf{P}_s\wedge\mathbf{V}_s\cdot\mathbf{H}_s}{c}\right\}\frac{1}{Y_s^{\frac{1}{2}}} = \frac{1}{Y_s^{\frac{1}{2}}}\left(\frac{\partial\chi}{\partial t_s} + \mathbf{V}_s\cdot\frac{\partial\chi}{\partial\mathbf{P}_s}\right) + \frac{1}{Y_s^{\frac{1}{2}}}\frac{d}{dt_s}(m_s c^2\xi_s^{\frac{1}{2}}).$$

Multiplying the last equation by $Y_s^{\frac{1}{2}}dt_s/dt$, and adding n similar equations, we get

$$\frac{d}{dt}\left[\sum_{s=1}^{n} m_s c^2\xi_s^{\frac{1}{2}}\right] + \frac{d\chi}{dt}$$

$$= [\text{rate of performance of electromagnetic work}].$$

The right-hand side may be reduced as in the previous chapter, and we get eventually as the grand energy equation

$$\frac{d}{dt}\left(\sum_{s=1}^{n} m_s c^2\xi_s^{\frac{1}{2}} + \chi + \Phi\right) = \sum_{s=1}^{n}\dot{\mathbf{V}}_s\cdot\frac{\partial\Phi}{\partial\mathbf{V}_s}\frac{dt_s}{dt}.$$

The left-hand side is to be interpreted as the sum of the mechanical, gravitational, and electromagnetic energies—all scalars—and the right-hand side as the negative rate of radiation. The epochs t_s may

be assumed connected with one another as functions of a parameter t by means of relations

$$X_1 = X_2 = \ldots = X_s = \ldots = X_n.$$

214. The charge e_s as a scalar. In the above analysis and in the preceding chapter, the scalars e_s have been interpreted as charges, since they reduce to the ordinary electrostatic definitions of charges when the corresponding particles are at rest and the epochs t_s coincide with t_0. With this definition of charge, the charge is the same whatever the observer chosen; charge is conserved under a transformation from any one fundamental observer to any other, just as mass and energy are conserved. An alternative view would be to regard the secular factor t_s/t_0 and the velocity denominator Y_s^{-1} or $(1 - V_s^2/c^2)^{-\frac{1}{2}}$ as absorbed into the definition of charge, putting $G_s = e_s(t_s/t_0)Y_s^{-\frac{1}{2}}$. This is more in accordance with the usage in current electrodynamics, where the measure of charge undergoes a transformation when the frame of reference is altered. But we shall see in the next chapter that the relativistic denominator plays an essential part in influencing the dynamics of a moving charge, and that it is best to retain the name charge for the actual invariant scalar e_s.

THE 'RADIUS' OF A POINT CHARGE

215. Interaction of two point-charges. The purpose of the present chapter is to apply the general electrodynamics developed in the two preceding chapters to the interaction of two point-charges. This is the general two-body problem of electrodynamics. But, as in gravitational theory, progress is made by first considering the simple 'one-body' or Keplerian problem, in which one of the bodies has so large a mass compared with the other's that it may be considered as fixed. We can then take the massive particle to coincide permanently with a fundamental particle of the substratum.

Accordingly we now consider two point-singularities P_1 and P_2, of charges e_1 and e_2, and masses m_1 and m_2, respectively, of which m_2 is large compared with m_1. We take P_2 to coincide with a fundamental particle. We require the motion of P_1 about P_2. Let O be the fundamental particle with which P_2 coincides. Then $\mathbf{P_2} = 0$, $\mathbf{V_2} = 0$, $Y_2 = 1$. Putting these values in the exact formulae for $\mathbf{E}$ and $\mathbf{H}$, (37) and (38) of Chapter XIII, we have

$$\mathbf{E_1} = \frac{e_2}{Y_1^3} \frac{\mathbf{P_1}}{|\mathbf{P_1}|^3}, \tag{1}$$

$$\mathbf{H_1} = \frac{e_2}{Y_1^3}\left(\tfrac{1}{2}\mathbf{V_1} \wedge \frac{\mathbf{P_1}}{|\mathbf{P_1}|^3}\right). \tag{2}$$

Since we are considering only the motion of e_1, we can omit the suffix 1 from all symbols except e_1. Then introducing (1) and (2) in formulae (16) and (16') of Chapter XIII, we have for the external force $(\mathbf{F}, F_t)$ on e_1

$$\mathbf{F} = \frac{e_1}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\frac{e_2}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\left(\frac{\mathbf{P_1}}{|\mathbf{P_1}|^3} + \frac{\mathbf{V}}{c} \wedge \left(\frac{1}{2}\frac{\mathbf{V}}{c} \wedge \frac{\mathbf{P_1}}{|\mathbf{P_1}|^3}\right)\right) + \frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{3}$$

$$F_t = \frac{e_1}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{e_2}{Y^{\frac{1}{2}}}\left(\frac{\mathbf{P_1}}{|\mathbf{P_1}|^3}\cdot\frac{\mathbf{V}}{c}\right) + \frac{c}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}). \tag{3'}$$

These simplify to

$$\mathbf{F} = \frac{e_1 e_2}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\frac{\mathbf{P} + \tfrac{1}{2}\mathbf{V} \wedge (\mathbf{V} \wedge \mathbf{P})/c^2}{|\mathbf{P}|^3} + \frac{\mathbf{V}}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}), \tag{4}$$

$$F_t = \frac{e_1 e_2}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\frac{\mathbf{P}\cdot\mathbf{V}}{c|\mathbf{P}|^3} + \frac{c}{Y^{\frac{1}{2}}}\frac{1}{Y^{\frac{1}{2}}}\frac{d}{dt}(m\xi^{\frac{1}{2}}). \tag{4'}$$

Introducing these now into the fundamental equations of motion, and cancelling on opposite sides the terms arising from rate of change of mass $m\xi^{\frac{1}{4}}$, we have

$$\frac{m\xi^{\frac{1}{4}}}{Y^{\frac{1}{4}}}\frac{d}{dt}\left(\frac{V}{Y^{\frac{1}{4}}}\right) = -\frac{m\xi^{\frac{1}{4}}}{X}\left(P-V\frac{Z}{Y}\right)+\frac{e_1 e_2}{t_0}\frac{Z}{Y^{\frac{1}{4}}}\frac{P+\frac{1}{2}V\wedge(V\wedge P)/c^2}{|P|^3}, \quad (5)$$

$$\frac{m\xi^{\frac{1}{4}}}{Y^{\frac{1}{4}}}\frac{d}{dt}\left(\frac{c}{Y^{\frac{1}{4}}}\right) = -\frac{m\xi^{2}}{X}\left(ct-c\frac{Z}{Y}\right)+\frac{e_1 e_2}{t_0}\frac{Z}{Y^{\frac{1}{4}}}\frac{P.V}{c|P|^3}. \quad (5')$$

We note that (5′) follows from (5) on scalar multiplication of (5) by V.

216. Remarks on the equations of interaction. Before we proceed to solve these equations, some comments on them may be made. In the first place, they should be exact. No terms have been omitted. In particular they do not require any correction for the supposed 'reaction of emitted radiation'. Any effect of this kind which might arise in a general electrodynamic situation is properly taken care of by the precise evaluation of (H, E) and the insertion of these field values in the Larmor-Lorentz ponderomotive force formula. This we have done. Moreover, there is no question of our having to connect the two epochs t_1 at e_1 and t_2 at e_2, for the equations have come out formally independent of t_2. The 't' in (5) and (5′) always stands for t_1. Thus there is no radiative interaction of e_1 and e_2 in this pure Keplerian two-particle problem; e_2 being at rest produces a static field at e_1.

Secondly, the equations (5) and (5′) have been formulated in t-measure, and hold good in the private Euclidean space of the observer at the origin using t-time. Moreover, (5) and (5′) are particular cases of Lorentz-invariant equations, in which all effects of change of mass with velocity have been fully taken into account, although $m\xi^{\frac{1}{4}}$, the mass of the moving particle carrying the charge e_1, appears outside the operator d/dt. The terms in $d(m\xi^{\frac{1}{4}})/dt$ have disappeared.

Thirdly, the electromagnetic terms contain a factor $Y^{-\frac{3}{2}}$. This arises as the product of three distinct factors $Y^{-\frac{1}{2}}$. One factor of this type occurs in each of E and H. A second arises from the need for the occurrence of the 4-vector $(V/Y^{\frac{1}{4}}, c/Y^{\frac{1}{4}})$ in the expression for the ponderomotive force. A third arises from the need for the invariant factor $Z/Y^{\frac{1}{4}}$ in the formula for external force F, necessary to remove a denominator Z in the energy-formula.

Fourthly, in (5) the magnetic term contains the factor $\frac{1}{2}$. This, the Thomas factor, is in our case a consequence of our modification

of the Biot–Savart formula for the magnetic effect of a moving charge. The physical interpretation of the term $\frac{1}{2}V \wedge (V \wedge P)/c^2|P|^3$ is that relative to the charge e_1 the massive particle with charge e_2, though really at local rest, seems to be moving with a speed V relative to e_1, and this creates at e_1 an apparent magnetic field; the mechanical effect of this magnetic field introduces a further factor V into the vector product.

It would be foreign to our method to deal with the motion by the use of the supposed 'laws' of conservation of energy and angular momentum. It is in fact precisely our object to inquire, by formulating the kinematic possibilities, how far these supposed 'laws' hold good as consequences of our equations of motion, and what forms they take. We therefore proceed to obtain certain first integrals of (5) and (5′), and compare them with the supposed conservation laws.

217. The energy equation. As often remarked previously, in our dynamics energy is a 4-scalar, not the fourth component of a vector. Hence equation (5′), the time-component equation corresponding to (5), though resembling an energy equation, is not really one. To obtain a genuine energy equation, we multiply (5) scalarly by

$$\frac{V}{Y^{\frac{1}{2}}} - P\frac{Y^{\frac{1}{2}}}{Z},$$

multiply (5′) by the corresponding time-component

$$\frac{c}{Y^{\frac{1}{2}}} - ct\frac{Y^{\frac{1}{2}}}{Z},$$

and subtract. After some reductions we obtain

$$\frac{d}{dt}(mc^2\xi^{\frac{1}{2}}) = \frac{e_1 e_2}{t_0}\frac{1}{Y^{\frac{1}{2}}}\left\{t(P.V) - P^2 + \frac{1}{2}\frac{(P \wedge V)^2}{c^2}\right\}\frac{1}{|P|^3}, \qquad (6)$$

a single factor $Y^{-\frac{1}{2}}$ surviving on the right-hand side.

It is important to verify that this result is in conformity with the general theory of Chapter XIII. This gives here the energy-formula

$$\frac{d}{dt}(mc^2\xi^{\frac{1}{2}} + \Phi) = \dot{V}.\frac{\partial\Phi}{\partial V}, \qquad (7)$$

where Φ, the electromagnetic energy, is given by

$$\Phi = \frac{1}{2}\frac{e_1 e_2}{ct_0}\left\{\frac{Z_1 Z_2 + Z_{12} Z_{21}}{Y_1^{\frac{1}{2}} Y_2^{\frac{1}{2}}(X_{12}^2 - X_1 X_2)^{\frac{1}{2}}}\right\}_{P_2=0, V_2=0} = \frac{e_1 e_2}{t_0}\frac{t - \frac{1}{2}(P.V)/c^2}{Y^{\frac{1}{2}}|P|}. \qquad (8)$$

Now (7) is equivalent to

$$\frac{d}{dt}(mc^2\xi^4) = -\frac{\partial\Phi}{\partial t} - \mathbf{V}\cdot\frac{\partial\Phi}{\partial\mathbf{P}}. \tag{9}$$

Some simple vector differentiation applied to (8) and inserted in (9) at once yields (6).

218. Transformation to τ-measure. Relation (6) may now be written in the form

$$Y^4\frac{d}{dt}(mc^2\xi^4) = -e_1 e_2 \frac{d}{dt}\left(\frac{t}{t_0|\mathbf{P}|}\right) + \frac{e_1 e_2}{t_0} \frac{\frac{1}{2}(\mathbf{P}\wedge\mathbf{V})^2}{c^2|\mathbf{P}|^3}. \tag{10}$$

So far, our analysis claims to be exact. But since $|\mathbf{P}| \ll ct$, the last term is utterly negligible compared with the other term on the right-hand side. Hence the equation is very closely

$$Y^4\frac{d}{dt}(mc^2\xi^4) = -e_1 e_2 \frac{d}{dt}\left(\frac{t}{t_0|\mathbf{P}|}\right). \tag{11}$$

We now transform this to τ-measure, by the transformation-formulae of Chapter VI. We know that

$$\xi^4 = \frac{d\tau}{d\sigma} = \frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}, \qquad |\mathbf{P}| = \frac{t}{t_0}|\mathbf{\Pi}|.$$

Further we know that

$$\mathbf{V} = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}\left(\frac{t}{t_0}\mathbf{\Pi}\right) = \frac{\mathbf{\Pi}}{t_0} + \frac{t}{t_0}\frac{d\mathbf{\Pi}}{dt} = \frac{\mathbf{\Pi}}{t_0} + \frac{d\mathbf{\Pi}}{d\tau} = \frac{\mathbf{\Pi}}{t_0} + \mathbf{v},$$

so that Y^4 may be replaced by $(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}$. Accordingly (11) becomes

$$(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}\frac{d}{d\tau}\left\{\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\right\} = -\frac{e_1 e_2}{mc^2}\frac{d}{d\tau}\left(\frac{1}{|\mathbf{\Pi}|}\right). \tag{12}$$

As is to be expected, the magnetic term plays no part in this, the energy-formula.

219. The 'energy'-integral. We can now conveniently put r for $|\mathbf{\Pi}|$. Integrating, we get

$$\log\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}} + \frac{e_1 e_2}{mc^2}\frac{1}{r} = \text{const.} \tag{13}$$

Call the constant of integration

$$\log\left(1 + \frac{W}{mc^2}\right)$$

where necessarily $W > -mc^2$. Then we have

$$\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}} = \left(1+\frac{W}{mc^2}\right)\exp\left(-\frac{e_1 e_2}{mc^2}\frac{1}{r}\right).\qquad(14)$$

When v is small compared with c, and r large compared with $e_1 e_2/mc^2$, this formula reduces approximately to

$$\tfrac{1}{2}mv^2 + \frac{e_1 e_2}{r} = W,\qquad(15)$$

so that in these circumstances W is the Newton-Coulomb energy, and it remains constant during the motion.

220. Variability of the energy. The exact expression for the energy in our theory is, however,

$$mc^2\xi^{\frac{1}{2}}+\Phi,$$

which by (8) is, very closely, in τ-measure,

$$\frac{mc^2}{(1-v^2/c^2)^{\frac{1}{2}}} + \frac{e_1 e_2}{r}\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}.\qquad(16)$$

Put this equal to $mc^2 + W'$. Then

$$W' = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}\left(mc^2+\frac{e_1 e_2}{r}\right)-mc^2,\qquad(17)$$

or, by (14)

$$W' = mc^2\left\{\left(1+\frac{W}{mc^2}\right)\left(1+\frac{e_1 e_2}{mc^2}\frac{1}{r}\right)\exp\left(-\frac{e_1 e_2}{mc^2}\frac{1}{r}\right)-1\right\}.\qquad(18)$$

When $r \gg |e_1 e_2|/mc^2$, this gives to a close approximation

$$W' = W$$

with neglect only of the square of $e_1 e_2/mc^2 r$. But as soon as r becomes comparable with $e_1 e_2/mc^2$, W' begins to differ from W, and, further, to vary with r. Though W is a constant of the motion, and the motion obeys (14), the actual energy is not a constant of the motion; there is in fact, in general, a fluctuation of energy as r varies. Expanding the exponential in (18) we get approximately

$$W' = W - \frac{1}{2}\frac{(e_1 e_2)^2}{mc^2}\frac{1}{r^2}.\qquad(19)$$

Hence if r is decreasing, W' is decreasing, and energy is being lost;

if r is increasing, energy is being gained. We can easily write down a formula for the rate of variation of energy, for by (7) this is

$$-\dot{\mathbf{V}}\cdot\frac{\partial\Phi}{\partial\mathbf{V}} = -\frac{e_1 e_2}{t_0}\frac{t-\frac{1}{2}(\mathbf{P}\cdot\mathbf{V})/c^2}{|\mathbf{P}|}\frac{\dot{\mathbf{V}}\cdot\mathbf{V}/c^2}{(1-\mathbf{V}^2/c^2)^{\frac{3}{2}}}$$

$$= -\frac{e_1 e_2}{t_0}\frac{t-\frac{1}{2}(\mathbf{P}\cdot\mathbf{V})/c^2}{|\mathbf{P}|}\frac{d}{dt}\left(\frac{1}{Y^{\frac{1}{2}}}\right). \qquad (20)$$

If the charge e_1 is approaching e_2 in virtue of motion under electrostatic attraction ($e_1 e_2 < 0$), $\mathbf{V}^2$ is increasing, Y is decreasing, $1/Y^{\frac{1}{2}}$ is increasing, and the rate of loss of energy is positive. If e_1 is receding from e_2, against electrostatic attraction, $\mathbf{V}^2$ is decreasing, Y is increasing, $1/Y^{\frac{1}{2}}$ is decreasing, and the rate of loss of energy is negative, i.e. energy is being absorbed. The gain or loss of energy effectively ceases as soon as r becomes large compared with $|e_1 e_2|/mc^2$.

221. These results are in disagreement with the supposed consequences of Maxwell's theory, according to which an accelerated or retarded electron radiates at a rate proportional to the square of its acceleration, i.e. at a rate essentially positive. But it is in agreement with the physical facts of absorption and radiation, namely that separation of an electron and a positively charged nucleus ion involves absorption of energy, whilst the near approach of an electron to a nucleus involves emission of energy. The changes of energy of the accelerated or retarded electron are always *reversible* on the present theory.

222. Emergence of a characteristic length-measure for an electron. The quantity $|e_1 e_2|/mc^2$ is a length. When e_1 and e_2 are taken to be equal to the charge on an electron, it becomes e^2/mc^2, which is the classical 'radius' of the electron, of the order of 10^{-13} cm. We see that the energy, properly speaking, ceases to be a constant of the motion as soon as the separation of the two charges becomes comparable with this length. In assuming that the departure of the energy from constancy is balanced by radiation or absorption, we are, strictly speaking, going beyond our kinematic formulation, for we do not know whether the total energy should be conserved or not; to speak of radiation in this connexion is a concession to current modes of thought, and to speak of absorption is illogical, since we have not brought into the picture any external sources of radiation. All we can say from this analysis with certainty is that the energy

as measured by the sum of the kinetic energy and potential energy is not conserved when the distance between the charges is comparable with $|e_1 e_2|/mc^2$. This is a consequence of the dynamics; the charges remain always as particle-singularities—they are not the limits of small volume charges.

223. Angular momentum. We proceed to consider angular momentum. We cannot of course assume that angular momentum is conserved.

Multiplying the equation of motion (5) vectorially by $\mathbf{P}$, we find

$$\frac{m\xi^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac{1}{2}}}\right) = \frac{m\xi^{\frac{1}{2}}}{X}\frac{Z}{Y}(\mathbf{P}\wedge\mathbf{V}) + \frac{e_1 e_2}{t_0}\frac{Z}{Y^{\frac{1}{2}}}\frac{\frac{1}{2}(\mathbf{P}\wedge\mathbf{V})(\mathbf{P}.\mathbf{V})}{c^2|\mathbf{P}|^3}. \quad (21)$$

We see that it is only the magnetic term which contributes to this equation; for the first two terms determine motion in the substratum alone. Dividing by $m\xi^{\frac{1}{2}}/Y^{\frac{1}{2}}$, it becomes

$$\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac{1}{2}}}\right) - \frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac{1}{2}}}\frac{Z}{X} = -\frac{\mathbf{P}\wedge\mathbf{V}}{Y^{\frac{1}{2}}}\frac{\frac{1}{2}e_1 e_2}{mc^2}\frac{X^{\frac{1}{2}}}{t_0}\frac{d}{dt}\left(\frac{1}{|\mathbf{P}|}\right).$$

Using $dX/dt = 2Z$, we see that the left-hand side becomes a complete differential on multiplication by $X^{-\frac{1}{2}}$. Thus

$$\frac{d}{dt}\left(\frac{\mathbf{P}\wedge\mathbf{V}}{X^{\frac{1}{2}}Y^{\frac{1}{2}}}\right) = -\frac{\frac{1}{2}e_1 e_2}{mc^2}\frac{\mathbf{P}\wedge\mathbf{V}}{X^{\frac{1}{2}}Y^{\frac{1}{2}}}\frac{X^{\frac{1}{2}}}{t_0}\frac{d}{dt}\left(\frac{1}{|\mathbf{P}|}\right).$$

The integral of this vector differential equation is

$$\frac{\mathbf{P}\wedge\mathbf{V}}{X^{\frac{1}{2}}Y^{\frac{1}{2}}} = \text{const. } \exp\left\{-\frac{\frac{1}{2}e_1 e_2}{mc^2}\int^t \frac{X^{\frac{1}{2}}}{t_0}\frac{d}{dt}\left(\frac{1}{|\mathbf{P}|}\right)dt\right\}. \quad (22)$$

At the present epoch t_0, the integral is very closely

$$\frac{t}{t_0}\frac{1}{|\mathbf{P}|}.$$

We now transform to τ-measure. The last-written expression is simply $1/|\Pi|$, which we call $1/r$. We have also

$$\frac{\mathbf{P}\wedge\mathbf{V}}{X^{\frac{1}{2}}Y^{\frac{1}{2}}} = \frac{\mathbf{P}\wedge\left(\dfrac{\mathbf{P}}{t}+\dfrac{d\Pi}{d\tau}\right)}{X^{\frac{1}{2}}Y^{\frac{1}{2}}} \sim \frac{\dfrac{t}{t_0}\left(\Pi\wedge\dfrac{d\Pi}{d\tau}\right)}{t(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}} = \frac{\Pi\wedge\mathbf{V}}{t_0(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}.$$

Hence we can write (22) in τ-measure in the form

$$\frac{m\Pi\wedge\dfrac{d\Pi}{d\tau}}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}} = \text{const. } \exp\left(-\frac{\frac{1}{2}e_1 e_2}{mc^2}\frac{1}{r}\right).$$

Multiplying this relation scalarly by $\mathbf{\Pi}$, we see that $\mathbf{\Pi}$ is perpendicular to a fixed vector, and therefore the motion lies in one plane. Using then plane polar coordinates (r, θ), the last relation becomes

$$\frac{mr^2\, d\theta/d\tau}{(1-v^2/c^2)^{\frac{1}{2}}} = H \exp\left(-\frac{\frac{1}{2}e_1 e_2}{mc^2}\frac{1}{r}\right), \qquad (23)$$

where H is a constant and

$$v^2 = \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2. \qquad (24)$$

224. Variability of angular momentum. We see that when r is large compared with $\frac{1}{2}|e_1 e_2|/mc^2$, the angular momentum

$$\frac{mr^2\, d\theta/d\tau}{(1-v^2/c^2)^{\frac{1}{2}}}$$

remains constant and equal to H. But when r becomes comparable with $\frac{1}{2}|e_1 e_2|/mc^2$, the angular momentum is no longer approximately constant but becomes a function of r. Again we see, as for energy, a departure from a conservation law as soon as r becomes comparable with $\frac{1}{2}|e_1 e_2|/mc^2$. This effect is directly traceable to the magnetic term in the interaction between the two charges.

225. Classical 'radius' of the electron. It was suggested by Bohr long ago that the classical integrals of energy and angular momentum in the Keplerian problem no longer hold good when the separation of the two charges becomes comparable with the classical 'radius' of the electron, though I am not aware that he ever embodied the idea in precise analysis, or obtained our modifications of the integrals, namely (14) and (23). In our work the effect ascribed to the apparent possession of a 'radius' by point singularities is a consequence of the occurrence in the equation of motion of appropriate factors $(1-v^2/c^2)^{-\frac{1}{2}}$ together with a proper calculation of the magnetic interaction, including the Thomas factor $\frac{1}{2}$. We have ascribed no structure to the charges, not even a spin. The 'radius'-effect is due to the formulation of equations of motion in accordance with the requirements of relativity. We have made no special hypotheses, and our laws of interaction reduce to the ordinary Coulomb expression (as regards the electric intensity) for low speeds. The interaction is strictly inverse-square, as is seen from equations (5) and (5').

226. Differing origins of variability of energy and of angular momentum. The present analysis shows a distinction between the 'energy' integral and the 'angular momentum' integral in that the former arises solely from the electrostatic interaction, the latter solely from the magnetic interaction. The former involves the length $|e_1 e_2|/mc^2$, the latter the length $\frac{1}{2}|e_1 e_2|/mc^2$. We shall now see the further consequences of this factor $\frac{1}{2}$.

227. Differential equation of the orbit. Let us seek the differential equation of the orbit of e_1 about e_2 in plane polar coordinates, as reckoned by an observer at e_2. Put for brevity

$$\frac{e_1 e_2}{mc^2}\frac{1}{r} = x. \tag{25}$$

Then integrals (14) and (23) may be rewritten in the forms

$$\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}} = \left(1+\frac{W}{mc^2}\right)e^{-x}, \tag{26}$$

$$\frac{mr^2\, d\theta/d\tau}{(1-v^2/c^2)^{\frac{1}{2}}} = He^{-\frac{1}{2}x}. \tag{27}$$

We wish to eliminate the time τ between these equations.

Dividing (27) by (26) we get

$$mr^2 d\theta/d\tau = \frac{H}{1+W/mc^2}e^{+\frac{1}{2}x}.$$

But

$$v^2 = \left(\frac{d\theta}{d\tau}\right)^2\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}.$$

Eliminating $d\theta/d\tau$ between the last two relations we have

$$v^2 = \frac{H^2}{m^2(1+W/mc^2)^2}\left\{\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}e^x,$$

or, substituting from r in terms of x from (25),

$$\frac{v^2}{c^2} = \frac{H^2c^2}{e_1^2 e_2^2}\left(1+\frac{W}{mc^2}\right)^{-2}\left\{x^2+\left(\frac{dx}{d\theta}\right)^2\right\}e^x.$$

Substituting for v^2/c^2 in (26) we now get

$$\left(1+\frac{W}{mc^2}\right)^{-2}e^{2x} = 1-\frac{H^2c^2}{e_1^2 e_2^2}\left(1+\frac{W}{mc^2}\right)^{-2}\left\{x^2+\left(\frac{dx}{d\theta}\right)^2\right\}e^x,$$

or

$$x^2+\left(\frac{dx}{d\theta}\right)^2 = \left\{\left(1+\frac{W}{mc^2}\right)^2 e^{-x}-e^x\right\}\frac{e_1^2 e_2^2}{H^2 c^2}. \tag{28}$$

Differentiating this relation with respect to θ and removing the factor $dx/d\theta$, we get

$$x + \frac{d^2x}{d\theta^2} = -\frac{1}{2}\frac{e_1^2 e_2^2}{H^2 c^2}\left\{\left(1 + \frac{W}{mc^2}\right)^2 e^{-x} + e^x\right\}. \tag{29}$$

This is a form of the differential equation of the orbit suitable for direct comparison with Newtonian mechanics.

228. Apparent repulsion or attraction between two charges. The last equation is in the form of the differential equation of a central orbit under a certain law of force. If f is the equivalent force of repulsion between the particle e_2 anchored at the origin and the particle e_1 at distance r, then the usual equations

$$m(\ddot{r} - r\dot{\theta}^2) = f,$$

$$mr^2\dot{\theta} = H,$$

give on putting $1/r = u$

$$u + \frac{d^2u}{d\theta^2} = -\frac{mf}{H^2 u^2}, \tag{30}$$

a well-known equation in the theory of central orbits.

But

$$u = \frac{mc^2}{e_1 e_2}x. \tag{31}$$

Hence (30) may be written

$$x + \frac{d^2x}{d\theta^2} = -\frac{m}{H^2}\left(\frac{e_1 e_2}{mc^2}\right)^3 \frac{f}{x^2}. \tag{32}$$

The actual differential equation of the orbit, (29), will coincide with the classical equation (32) if for f we take

$$f = +\frac{1}{2}\frac{m^2 c^4}{e_1 e_2}x^2\left\{\left(1 + \frac{W}{mc^2}\right)^2 e^{-x} + e^x\right\},$$

or, reverting to the r-notation,

$$f = +\frac{1}{2}\frac{e_1 e_2}{r^2}\left\{\left(1 + \frac{W}{mc^2}\right)^2 e^{-(e_1 e_2/mc^2)(1/r)} + e^{(e_1 e_2/mc^2)(1/r)}\right\}. \tag{33}$$

This, then, is the *apparent force* of repulsion between the charge e_1 and e_2 which will give the same differential equation of the orbit on non-relativistic Newtonian mechanics as the actual differential equation on our relativistic mechanics.

229. Particular cases. We have the following particular cases of (33). When $|W| \ll mc^2$, and $r \gg |e_1 e_2|/mc^2$, we have

$$f \sim \frac{e_1 e_2}{r^2}, \tag{34}$$

the ordinary Coulomb law. When r is so small that it is comparable with $|e_1 e_2|/mc^2$, we need the two-term formula (33). When r is smaller still, and indeed small compared with $|e_1 e_2|/mc^2$, one of the two exponential terms is very large and the other very small, depending on the sign of the product $e_1 e_2$. When e_1 and e_2 are of the same sign, so that $e_1 e_2 > 0$, the repulsion for $r \ll e_1 e_2/mc^2$ is very closely

$$f \sim \frac{1}{2} \frac{e_1 e_2}{r^2} e^{(e_1 e_2/mc^2)(1/r)}, \tag{35}$$

and thus increases enormously rapidly as r decreases still further. Like charges thus behave as impenetrable into one another's neighbourhoods at distances much less than $e_1 e_2/mc^2$, in spite of the fact that each is a mere point singularity. When e_1 and e_2 are of opposite signs, the repulsion changes to an attraction, and if $r \ll |e_1 e_2|/mc^2$, it is very closely

$$f \sim \frac{1}{2} \frac{e_1 e_2}{r^2} \left(1 + \frac{W}{mc^2}\right)^2 e^{-(e_1 e_2/mc^2)(1/r)}, \tag{36}$$

which again increases enormously (numerically) as r decreases.

Thus the Coulomb inverse square law changes into something quite different, effectively, as r passes through the value $|e_1 e_2|/mc^2$. Strictly speaking, the exact Coulomb law holds at all distances, however small; the modifications are due to effects of dynamics.

230. Interpretation of the constant W. The interpretation of the constant W on Newtonian mechanics can be found as follows. Let U be the energy constant of the Newtonian equivalent orbit. Then, integrating equation (30), we get

$$\frac{1}{2}\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = -\frac{m}{H^2}\left(\int_0^u \frac{f}{u^2} du - U\right), \tag{37}$$

where

$$mr^2\left(\frac{d\theta}{dt}\right) = H. \tag{38}$$

Thus the previous equation is equivalent to

$$\tfrac{1}{2}m\left\{r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right\} = U - \int_0^u \frac{f}{u^2} du.$$

Since f is the force of repulsion,

$$\int_0^u \frac{f}{u^2} du = \int_r^\infty f \, dr$$

is the potential energy of the configuration at any instant, and so U is the sum of the kinetic and potential energies, calculated classically. Introducing (33) for f into (37) we get

$$\frac{1}{2}\frac{H^2}{m}\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\} = U - \int_0^u \tfrac{1}{2}e_1 e_2\left\{\left(1+\frac{W}{mc^2}\right)^2 e^{-(e_1 e_2/mc^2)u} + e^{(e_1 e_2/mc^2)u}\right\} du$$

$$= U+\tfrac{1}{2}mc^2\left\{\left(1+\frac{W}{mc^2}\right)^2 e^{-(e_1 e_2/mc^2)u} - e^{(e_1 e_2/mc^2)u}\right\} -$$

$$-\tfrac{1}{2}mc^2\left\{\left(1+\frac{W}{mc^2}\right)^2-1\right\}.$$

This may be written

$$x^2+\left(\frac{dx}{d\theta}\right)^2 = \frac{2m}{H^2}\left(\frac{e_1 e_2}{mc^2}\right)^2\left\{U+\tfrac{1}{2}mc^2\left(1+\frac{W}{mc^2}\right)^2(e^{-x}-1)-\tfrac{1}{2}mc^2(e^x-1)\right\}$$

$$= \frac{e_1^2 e_2^2}{H^2 c^2}\left[\left\{\left(1+\frac{W}{mc^2}\right)^2 e^{-x}-e^x\right\}+\frac{U}{\tfrac{1}{2}mc^2}-\left(1+\frac{W}{mc^2}\right)^2+1\right]. \quad (39)$$

Comparing this with (28), we see that

$$U = \tfrac{1}{2}mc^2\left\{\left(1+\frac{W}{mc^2}\right)^2-1\right\}, \tag{40}$$

or

$$U = W\left(1+\frac{1}{2}\frac{W}{mc^2}\right). \tag{41}$$

As was to be expected, when $W \ll mc^2$, $U \sim W$. We see that

$$\left(1+\frac{W}{mc^2}\right)^2 = 1+\frac{U}{\tfrac{1}{2}mc^2}, \tag{42}$$

and hence $U > -\tfrac{1}{2}mc^2$. The apparent modification of the Coulomb attraction, (36) with $e_1 e_2 < 0$, may now be written

$$f \sim \frac{1}{2}\frac{e_1 e_2}{r^2}\left(1+\frac{U}{\tfrac{1}{2}mc^2}\right)e^{-(e_1 e_2/mc^2)(1/r)}, \tag{43}$$

where now U is the Newtonian energy of the orbit calculated non-relativistically.

231. Magnetic factor $\frac{1}{2}$. The forms of these various formulae depend intimately on the coefficient $\frac{1}{2}$ in the magnetic term for the interaction. This factor $\frac{1}{2}$ appears in the exponential in formula (27), and it governs all the subsequent analysis. Were it not present, we should not have arrived at the remarkable formula for the apparent modification of the Coulomb law given by (33). This is different in form from the modification empirically suggested on other grounds by Yukawa, and different again from the modification suggested by Eddington. Formula (33) shows at once why the electron behaves as having an apparent radius e^2/mc^2.

It is therefore incumbent on us to scrutinize in some detail the modification of the Biot and Savart law for the magnetic field produced by a moving charge at another moving charge, to which we were led in Chapter XIII.

232. Reconciliation with classical Biot and Savart law in certain instances. We saw that the magnetic field H_1 at a charged particle e_1 moving with velocity V_1 relative to a fundamental observer, as due to a second moving charge e_2 moving with V_2, distant r away, is given by

$$H_1 = \frac{e_2}{c}\tfrac{1}{2}(V_1+V_2)\wedge\frac{r}{|r|^3},\tag{44}$$

where $r = r_1 - r_2$. Consider now the magnetic field at a point P_1 of a current-carrying conductor C_1, due to a second current-carrying conductor C_2. Conductors C_1 and C_2 are supposed to be electrostatically neutral; any volume element of either is occupied at any instant by equal and opposite charges. The magnetic field at the point P_1 of C_1 will be the superposition of the microscopic magnetic fields in the volume element surrounding P_1. This volume element will contain, say, charges $+e_1$ and $-e_1$ moving respectively with velocities V_1 and V_1' relative to the observer. The macroscopic field will be the sum of the fields associated with $+e_1$ and $-e_1$. Now consider the contribution to this field arising from the charges $+e_2$ and $-e_2$ occupying a volume element surrounding P_2, a point in conductor C_2, these charges moving respectively with velocities V_2 and V_2', relative to the observer. This contribution can be analysed into the contributions from the pairs

$$(+e_1, +e_2),\ (-e_1, +e_2),\ (+e_1, -e_2),\ (-e_1, -e_2).$$

Writing down the microscopic fields associated with these pairs according to the formula quoted above, we get on superposition

$$\left[\frac{1}{2}\frac{e_2}{c}(V_1+V_2)+\frac{1}{2}\frac{e_2}{c}(V_1'+V_2)-\frac{1}{2}\frac{e_2}{c}(V_1+V_2')-\frac{1}{2}\frac{e_2}{c}(V_1'+V_2')\right]\wedge\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

which comes to $\qquad\qquad \dfrac{e_2}{c}(V_2-V_2')\wedge\dfrac{\mathbf{r}}{|\mathbf{r}|^3}.$ $\qquad$ (45)

But this is precisely what would be calculated from the *classical* form of the Biot and Savart law, which takes no account of the velocity of the test-charge e_1. Relation (45), in fact, attributes the magnetic field at P_1 entirely to the effects of the convection currents due to $+e_2$ at P_2 moving with V_2 and $-e_2$ at P_2 moving with $+V_2'$, without any factors $\frac{1}{2}$.

Consider next the field at P_1 due to the same current-carrying conductor C_2, when measured by the charges in motion constituting an electrostatically neutral magnet at P_1. With the same notation, the time-mean velocities V_1 and V_1' will be zero, and we shall get the same result.

Next, consider the field at P_1, as measured by an electrostatically neutral magnet at P_1 when originated by a single charge e_2 at P_2 moving with speed V_2. With the same notation, the macroscopic field at P_1 is now

$$\left[\frac{1}{2}\frac{e_2}{c}(V_1+V_2)+\frac{1}{2}\frac{e_2}{c}(V_1'+V_2)\right]\wedge\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

and since here $V_1 = 0$, $V_1' = 0$, the mean field is

$$e_2\frac{V_2}{c}\wedge\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

which is again what would be given by a crude application of the classical Biot and Savart law.

In all these cases the result is the same as that calculated classically. But a difference occurs when the magnetic field at P_1 is measured by a single moving charge at P_1, as in the case considered in the earlier part of this chapter. For example, suppose the field arises from an electrostatically neutral current-carrying conductor C_2 at P_2. The macroscopic field is then

$$\left[\frac{1}{2}\frac{e_2}{c}(V_1+V_2)-\frac{1}{2}\frac{e_2}{c}(V_1+V_2')\right]\wedge\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

or $\qquad\qquad \dfrac{1}{2}\dfrac{e_2}{c}(V_2-V_2')\wedge\dfrac{\mathbf{r}}{|\mathbf{r}|^3}.$

This has the merit of being independent of the speed V_1 of the test-charge e_1, so that the case corresponds to the existence of an objective magnetic field at P_1; but it is one-half of that calculated by a crude application of the classical Biot and Savart law. As this factor $\frac{1}{2}$ has been seen to play an essential part in the calculations describing the interaction of two point-charges—calculations which result in the apparent attribution of a radius e^2/mc^2 to an electron in the vicinity of a proton—I hold that this factor $\frac{1}{2}$ and the occurrence of the velocity of the test-charge in the microscopic formula are really required.

233. In the next chapter we shall tentatively apply the new integrals of energy and angular momentum obtained in this chapter to a simple model atom, following the original method due to Bohr.

ATOMIC SYSTEMS. THE NEUTRON

234. Keplerian atomic problem. We shall now take e_2 to be the charge on a proton, namely $e_2 = +e$. We consider the possible orbits of a negatively charged electron, $e_1 = -e$, in the vicinity of this proton. It is true that wave-mechanics replaces in the end the concept of a point-electron by the concept of a probability-distribution of electric charge, but the results of applying wave-mechanics to atomic systems coincide in many cases with the results obtained by treating electrons as particles. As in this book we are pushing the concept of particle to its utmost limit, and as the concepts and assumptions of wave-mechanics are foreign to the class of ideas developed in this book, we shall pursue our brief study of atomic systems by the methods of Bohr-Sommerfeld, in which quantization is applied directly to the dynamical variables.

235. Restatement of integrals of energy and angular momentum. We begin by restating our integrals of energy and angular momentum for the Keplerian problem of an electron of small mass m in motion in the vicinity of a proton whose mass M is so large relative to it that it may be taken to be at rest. We shall further, as before, take it to coincide with a fundamental particle. To avoid confusion with the exponential base e, we shall write $+\epsilon$ for the charge on a proton, $-\epsilon$ for the charge on an electron. We shall then change the sign of x, and instead of writing x for $e_1 e_2/mc^2 r$ we shall write

$$x = +\frac{\epsilon^2}{mc^2}\frac{1}{r}. \tag{1}$$

The integrals of energy and angular momentum may now be written, from (26) and (27) of Chapter XV,

$$\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}} = \left(1+\frac{W}{mc^2}\right)e^{+x}, \tag{2}$$

$$\frac{mr^2 d\theta/d\tau}{(1-v^2/c^2)^{\frac{1}{2}}} = He^{+\frac{1}{2}x}. \tag{3}$$

The differential equation of the orbit, obtained by eliminating $d\tau$ between these, takes the form, by (28) of Chapter XV,

$$x^2+\left(\frac{dx}{d\theta}\right)^2 = \left[\left(1+\frac{W}{mc^2}\right)^2 e^{+x}-e^{-x}\right]\frac{\epsilon^4}{H^2 c^2}. \tag{4}$$

236. When $r \gg \epsilon^2/mc^2$, so that x is small, (2) reduces to

$$\tfrac{1}{2}mv^2 - \frac{\epsilon^2}{r} = W,\tag{2'}$$

and (3) reduces to

$$mr^2\frac{d\theta}{d\tau} = H.\tag{3'}$$

The Bohr theory of the hydrogen atomic orbits follows on taking H to be a multiple of $h_0/2\pi$. It is outside our task in this book to develop a quantum theory of the atom from first principles, and we shall therefore take over the quantization of H,

$$H = \frac{nh_0}{2\pi},\tag{5}$$

unchanged. It should be noticed that in this formula h_0 is a constant. The corresponding formula in t-measure is obtained by equating the angular momentum to a multiple of $(h_0/2\pi)(t/t_0)$. The present h_0 is therefore the h_0 of Chapter VIII, whose value is proportional to the normalization constant t_0.

237. Fine-structure constant. The differential equation of the orbits, (4) above, may now be written

$$x^2 + \left(\frac{dx}{d\theta}\right)^2 = \left[\left(1 + \frac{W}{mc^2}\right)^2 e^x - e^{-x}\right]\frac{\alpha^2}{n^2},\tag{6}$$

where

$$\alpha = \frac{2\pi\epsilon^2}{h_0 c},\tag{7}$$

This pure number α is the 'fine-structure constant', and according to Eddington takes the value 1/137. That α is a pure number in spite of the secular dependence of h_0 on t_0 follows from equation (9), Chapter XIV, according to which the product of two charges $e_r e_s$, in particular the square ϵ^2, varies secularly proportionally to t_0. The ratio ϵ^2/h_0 is thus independent of t_0, as well as of t.

238. Circular orbits. The apses of the orbits whose differential equation is (6) are given by putting $dx/d\theta = 0$. The corresponding values of x are therefore roots of the equation

$$f(x) \equiv \left(1 + \frac{W}{mc^2}\right)^2 e^x - e^{-x} - \frac{x^2 n^2}{\alpha^2} = 0.\tag{8}$$

For a circular orbit, two adjacent apses must coincide, and

therefore the radii of the possible circular orbits are also the roots of the equation

$$f'(x) \equiv \left(1 + \frac{W}{mc^2}\right)^2 e^x + e^{-x} - 2x\frac{n^2}{\alpha^2} = 0. \tag{9}$$

Equations (8) and (9) form a simultaneous pair for the two unknowns x (giving the radius of a circular orbit) and W, the corresponding constant which occurs in the 'energy-integral'. Eliminating W between (8) and (9), we get

$$e^{-x} - \frac{n^2}{\alpha^2}(x - \tfrac{1}{2}x^2) = 0. \tag{10}$$

When x has been found as a solution of (10), the corresponding value of W is given by

$$1 + \frac{W}{mc^2} = \frac{n^2}{\alpha^2} x(1 - \tfrac{1}{4}x^2)^{\frac{1}{2}} = e^{-x}\left(\frac{1 + \tfrac{1}{2}x}{1 - \tfrac{1}{2}x}\right)^{\frac{1}{2}}. \tag{11}$$

The energy W' itself is given, by (18) Chapter XV, by

$$\begin{aligned} W' &= mc^2\left[\left(1 + \frac{W}{mc^2}\right)(1-x)e^x - 1\right] \\ &= mc^2\left[(1-x)\left(\frac{1 + \tfrac{1}{2}x}{1 - \tfrac{1}{2}x}\right)^{\frac{1}{2}} - 1\right], \end{aligned} \tag{12}$$

239. Location of roots giving apses. Consider now the roots of (10). The location of the roots is governed by the circumstance that α, the fine-structure constant, is small compared with unity. It follows that the roots of

$$x - \tfrac{1}{2}x^2 = \frac{\alpha^2}{n^2}e^{-x}$$

are close to the roots of

$$x - \tfrac{1}{2}x^2 = 0,$$

namely $\qquad\qquad x = 0 \quad$ and $\quad x = 2$.

There are in fact just two roots of (10).

240. Circular orbits of the hydrogen atom. The root near $x = 0$ is found to have the expansion

$$x \sim \frac{\alpha^2}{n^2}\left[1 - \frac{1}{2}\frac{\alpha^2}{n^2} + \frac{1}{2}\frac{\alpha^4}{n^4} + \cdots\right]. \tag{13}$$

The leading term gives the corresponding series of radii

$$r = \frac{\epsilon^2}{mc^2}\frac{1}{x} \sim \frac{\epsilon^2}{mc^2}\frac{n^2}{\alpha^2} = \frac{n^2 h_0^2}{4\pi^2 m\epsilon^2}, \tag{14}$$

which for n integral are the Bohr circular orbits of hydrogen. We note that the measure of r is proportional to t_0.

The corresponding value of W is given by

$$W = mc^2(-\tfrac{1}{2}x + \tfrac{1}{8}x^2 + ...)$$
$$= -\tfrac{1}{2}mc^2\frac{\alpha^2}{n^2}\left(1 - \frac{3}{4}\frac{\alpha^2}{n^2}\right), \tag{15}$$

and the actual energy W' by (12) is given by

$$W' = mc^2(-\tfrac{1}{2}x - \tfrac{3}{8}x^2 + ...)$$
$$= -\tfrac{1}{2}mc^2\frac{\alpha^2}{n^2}\left(1 + \frac{1}{4}\frac{\alpha^2}{n^2} + ...\right).$$

The leading term gives

$$W' = -\tfrac{1}{2}mc^2\frac{\alpha^2}{n^2} = -\frac{2\pi^2\epsilon^4 m}{n^2 h_0^2},$$

which is just the Bohr energy for the n-quantum state. Thus the circular orbits $x = \alpha^2/n^2$ correspond to the Bohr circular orbits of the hydrogen atom.

241. Root near $x = 2$. Put $x = 2 - \delta$. Then the equation for δ is

$$\delta - \tfrac{1}{2}\delta^2 = \frac{\alpha^2}{n^2}e^{-2+\delta},$$

of which the root near $\delta = 0$ is

$$\delta = \frac{\alpha^2}{n^2}e^{-2}\left[1 + \frac{3}{2}\frac{\alpha^2}{n^2}e^{-2} + ...\right].$$

The corresponding radius is

$$r = \frac{\epsilon^2}{mc^2}\frac{1}{2-\delta} \sim \frac{1}{2}\frac{\epsilon^2}{mc^2}\left[1 + \frac{1}{2}\frac{\alpha^2}{n^2}e^{-2} + ...\right].$$

Since α^2 is very small, the radii for different values of n are very close together, and crowd round the value

$$r \sim \frac{1}{2}\frac{\epsilon^2}{mc^2}.$$

This r is again proportional to t_0.

242. The neutron. This system, of an electron in very close proximity to a proton, and moving with a speed close to c, in one or other of a set of circular orbits of only slightly different radii, may be provisionally identified with the *neutron*. For at distances

comparable with the radius of the normal atom, it will appear as if electrostatically neutral. Its radius $\frac{1}{2}\epsilon^2/mc^2$ has the numerical value $1\cdot4\times10^{-13}$ cm.

The value of the constant W for this combination is found to be

$$W \sim mc^2\left(2\frac{n}{\alpha}e^{-1}-1\right),$$

and is positive. The value of the energy W' is found to be

$$W' \sim -mc^2\left(2\frac{n}{\alpha}e^{+1}+1\right),$$

and is negative. The various excited states of this combination, though of approximately the same radius, have very different binding energies. It must be remembered that W, though not in general the energy, is a constant of the motion, whilst W', the energy, is not a constant of the motion save for circular orbits, for which x and r remain constant.

The differential equation of the orbits, namely (6), can be written in the form

$$\frac{n^2}{\alpha^2}\left(\frac{dx}{d\theta}\right)^2 = y(x),$$

where

$$y(x) \equiv \left(1+\frac{W}{mc^2}\right)^2 e^x - e^{-x} - \frac{n^2}{\alpha^2}x^2.$$

The integrated orbits are therefore

$$\theta + \text{const.} = \frac{n}{\alpha}\int^x \frac{dx}{y^{\frac{1}{2}}}.$$

243. Instability of the neutron. The configurations of these orbits, as is well known, depend on the distribution of zeros of $y(x)$, which depend in a somewhat complicated way on the values of W and n/α. Graphs showing the location of these zeros have been published in *Phil. Mag.*, Ser. 7, vol. **34**, p. 251, 1943. It is there shown that for

$$0 > \frac{W}{mc^2} > -\frac{1}{2}\frac{\alpha^2}{n^2},$$

the loci $y = y(x)$ have two zeros separated by the repeated zero corresponding to the Bohr circular orbits of hydrogen, together with a third zero for which $x > 2$. In between the first two zeros, $y(x)$ is positive, whence it can be shown that the Bohr circular orbits of the hydrogen atom are stable, the zeros separated by the repeated zero giving the apses of the non-circular orbits. Between the second

and third zeros, $y(x)$ is negative, and no closed orbits exist. As W increases, the first orbits which appear are the circular orbits corresponding to the repeated root $x \sim 2$, for which

$$\frac{W}{mo} = -1 + 2\frac{n}{\alpha}e^{-1}.$$

For W still greater, $y(x)$ is positive for all x, and the orbits pass from infinity to the nucleus, spending, however, for W just larger than the last-mentioned value, a very large amount of their time in the neighbourhood of $x \sim 2$, which thus acts as a sort of potential barrier. It follows that the circular orbits near $x \sim 2$ are unstable to increases of W. Hence, on the present analysis, in which we identify the circular orbits $x \sim 2$ with the neutron (regarded as a close combination of a proton and an electron), the neutron is essentially unstable, though it may have a transient existence for an appreciable time. A neutron may decompose into a free proton and free electron, or give rise to the mutual annihilation of the constituent proton and electron, depending on the sense of direction of the electron's orbit, whether to or away from the nucleus. Along such paths, although the constant of integration W (which coincides very nearly with the actual energy W') remains constant, the actual energy W', as calculated on the present theory, varies. Energy is thus not conserved. As the orbital electron passes outwards from the vicinity of the neutron circular orbits to freedom, W' passes from a large negative value to the positive value corresponding to the kinetic energy of the freed electron, and energy is created. If, on the other hand, the orbital electron passes inwards to the nucleus, moving theoretically from freedom at infinity and spiralling many times round the nucleus in the neighbourhood of $x \sim 2$, positive kinetic energy is transformed into negative binding energy, and ultimately disappears. It will of course be ultimately replaced by the energy value of the masses disappearing, which should reappear as radiation.

244. Possibility of indefinite generation of energy. Further details will be found in the paper in the *Phil. Mag.* previously cited. Here it is sufficient to emphasize that when the energy W' is not a constant of the motion, and so energy is not conserved, it is desirable not to be entrapped by conventional phraseology, and to abstain from asking, Where does the energy go to or come from? In the deep interiors of stars, where owing to the high central densities electrons

may have a chance of finding themselves in such close vicinity to protons that they will form neutrons, the neutrons formed may be regarded, by their decomposition again into free protons and free electrons, as generators of energy. They would constitute, moreover, an inexhaustible source of energy, for the liberated electron and proton could under favourable circumstances re-form a neutron, and the cycle would recommence. It is to be remembered that this phenomenon of energy-generation, on the present theory, only sets in when electron and proton are brought within a distance ϵ^2/mc^2 of one another.

245. Discussion. In this chapter the electro-magnetic theory which we have developed on the basis of kinematic relativity has been pushed to its logical extremes, and perhaps farther than is justified. Much more consideration of its fundamentals is evidently desirable. Nevertheless it is extremely suggestive and promising that the modifications introduced into classical theory by our revised dynamics and electrodynamics all centre round effects predominant at separation-distances ϵ^2/mc^2. This length is not an arbitrary assumption of the theory, as it is in some other current modifications of electro-magnetic theory. On the contrary it is a length forced on our attention by the theory itself. The usual integrals of energy and angular momentum have appeared in a form which shows that energy and angular momentum are strictly conserved only when the separation distance of a proton and electron is large compared with this length ϵ^2/mc^2, as originally forecast by Bohr. But the emergence of this length has nothing to do, in our theory, with the supposed volume-occupation of a charge in some versions of the theory of electrons. On our theory, charges remain throughout as strictly point-singularities, and not as the limits of small volume-distributions. They attract and repel one another, at all distances, with strictly Coulomb inverse-square forces when at rest. These forces are modified by relativity factors for charges in motion, and these cause a change in the dynamics. The possibility of a second set of circular orbits for the electron in interaction with a proton was an entirely unsought consequence of the general theory, and it is difficult to resist a provisional identification of such configurations with the neutron, whose exact status in contemporary physics is still very doubtful.

GENERAL OUTLOOK

246. Survey of the theory. In this concluding chapter I propose to take a general view of the results of applying the methods developed in this book. These methods are described by the general title of 'Kinematic Relativity' because they are based essentially on the idea of relative motion and because they do not presuppose the existence of any particular system of dynamics. If the reader has had the patience to follow the various detailed investigations given in the foregoing, he may still have lost sight of the broad sweep of ideas of which they are the offspring. Let us therefore stand at some distance from the detailed results and survey the achievements of the theory.

247. Expansion of the universe. The theory originated in an attempt to give a rational account of the phenomenon of the expansion of the universe. Putting every nucleus of an extra-galactic nebula on the same footing, it shows that radially outward recession is the inevitable behaviour of a system of nebulae. This aspect of kinematic relativity was more particularly the theme of the volume to which the present work is a sequel. Here it is sufficient to recall that 'space' is not a physical attribute of the universe, but is a mode of description of phenomena which is at the disposal of the observer; and when, for purposes of simplicity, an observer has adopted a private infinite Euclidean space with which to describe the totality of things, he is bound to conclude that a collection of nebulae will ultimately suffer dispersion, and so mutual recession. When he imposes the further condition that no member of the collection is to be in a favoured situation, he will conclude further that the population of the system must be infinite, but (in his private Euclidean space) must appear to occupy a finite volume. And since the experienced density must be everywhere finite, the boundary of the apparent volume occupied by the universe must be the locus of limiting points at which the density *tends* to infinity. There is no actual paradox involved. Towards the boundary of the system the speed of recession must tend to the velocity of light, and the Lorentz contraction resulting means that there is room for the infinity of

members. The singularity at the boundary is in fact the consequence of the initial singularity which started the system into existence. For it is an essential feature of the theory that every equation in it makes explicit reference, by mention of the age t reckoned from the natural origin of time, to the creation of the system *in time*. The natural zero of time is in fact the epoch of this creation. Every assertion about the contents of the system involves the mention of this age t, and so takes note of the circumstance that the system was once *created*. This is the first definite point of departure from contemporary physics and from Einstein's theory of relativity. Contemporary physics takes the universe as given, and however far it traces backwards the past history of the universe, it has always a still infinite tract of time behind that, at its disposal. Contemporary physics is of course accustomed to assign a period of two to three thousand million years as the age of the universe, but the mere assigning of this period is a self-stultifying action unless it really means that *time*, as we understand it, *was not*, before then. Thus contemporary physics suffers from an inherent internal contradiction in using simultaneously a time variable capable of infinite negative values and a definite value of that time variable as being the epoch of the beginning of things.

248. Creation of the universe as a point-singularity. If the universe was created, so that time only runs from the epoch of creation, then it is difficult not to conclude also that the universe was created as a point-singularity. If it were to be supposed created 'all at once' as occupying a finite volume, then this would mean that a meaning could be attached to saying that events at two points spatially separated could be absolutely simultaneous. This is contradicted by Einstein's relativity. The origination of the universe at a point-singularity has been developed independently by Lemaître, who calls the point-singularity a 'super-atom'. It is, however, undesirable to attempt to describe the state of things *at* $t = 0$, for such a state of things must necessarily transcend ordinary experience; we can only hope to describe the state of things *after* $t = 0$.

249. The substratum. These properties of the expanding universe are given their simplest expression in the model we have called the 'substratum'. In this model the nuclei of the extra-galactic nebulae are represented by a cloud of particles, called 'fundamental particles',

mutually separating, and the scale of time is chosen so that the relative velocities are uniform. The fundamental particles then provide a natural set of frames of reference, which are equivalent to one another. Each fundamental particle is the centre of symmetry of all the remainder, in its own frame of reference. The observer associated with each fundamental particle can describe the whole system in terms of his private Euclidean space. In this private Euclidean space the system occupies the interior of an expanding sphere, whose surface moves radially outwards with the speed of light. Inside this sphere, each fundamental particle has a velocity proportional to its distance from the centre, and the density of particles is locally homogeneous near the centre but increases steadily outwards. At each locality in the system, the density steadily decreases in time, owing to the expansion.

250. Motion of a free test-particle. The question then arose as to the motion of an arbitrary free test-particle in the presence of the substratum. This is not a problem peculiar to Kinematic Relativity; it arises in any scheme of physics. In the Newtonian scheme it is solved by positing the axiom that the motion of a free particle is one of uniform velocity; but the Newtonian scheme does not identify the frames of reference in which this uniform velocity subsists; it simply calls such frames 'inertial frames'. In the Einsteinian scheme, a free test-particle describes a geodesic in space-time, but the scale of time is not identified. In kinematic relativity, the frames of reference to which motions are referred are given by the fundamental particles and the scale of time is chosen to be such that these are in uniform relative motion. The equation of motion of a free test-particle in the substratum is found in the first instance to within an arbitrary scalar function. The arbitrariness of this scalar function corresponds to the fact that the same equation of motion holds good in the presence of certain generalizations of the substratum, here called statistical systems, and it is necessary to discover arguments which eliminate these generalizations or rather reduce them to the particular non-statistical, hydrodynamical system which is the substratum. These arguments reduce the arbitrary function $G(\xi)$ originally entering into the acceleration-formula for a free particle to a definite constant, namely $G(\xi) \equiv -1$, and so make the acceleration-formula determinate.

251. Identification of the time-scale. But the acceleration-formula so found is non-Newtonian in form. It follows that the scale of time employed in its derivation is not the Newtonian scale. But it becomes locally Newtonian for small velocities on taking as new time-variable the logarithm of the former time-variable multiplied by a normalization constant t_0 equal to the present age of the universe on the t-scale. The time-variable here referred to is the actual clock-reading of the clock supposed carried by a fundamental observer, which must be distinguished from the time-coordinate of a distant event. The clocks of the fundamental observers must thus be all regraduated.

252. The two scales of time. Since in Kinematic Relativity length-measures are made to depend on time-measures, the regraduation of clocks affects all length-measures and so all space-coordinates as well as time-coordinates. It also affects all velocities, changing all velocities into velocities relative to the fundamental particles with which the particles concerned momentarily coincide. This of course reduces the fundamental particles themselves to relative rest, and the substratum becomes relatively stationary. The new scale is called the τ-scale.

253. Much misunderstanding has been caused to critics of the theory by the occurrence of these two scales of time. It has been overlooked that they constitute two distinct languages, and the statement of any property of the system can be made in either language. The epoch of creation, $t = 0$, becomes on the τ-scale $\tau = -\infty$. It thus appears that the paradox into which contemporary physics is led in discussing the age of the universe, to which allusion has been made above, is due to the confusion of the two scales of time. Until we know which scale of time is being used in any given context, it is impossible to give an unambiguous meaning to any assertion involving time. Statements involving Newtonian mechanics, or Newtonian gravitation, are usually made in terms of the τ-scale. Statements involving the Lorentz-formulae, photons, or the electro-magnetic theory of Maxwell involve the t-scale. It should be remembered that there are different τ-scales corresponding to different values of the normalization-constant t_0; the t-scale is the absolute scale, as in it the acceleration formula involves no constant but c; the acceleration formula in τ-measure, in its exact form, involves mention in addition

of the 'constant' t_0, the choice of which is arbitrary, but whose physical meaning is the age of the universe at which the graduations of the two scales are made to coincide.

254. Coincidence at a point. The frivolous objection has been made to the idea of regraduation that it would be possible to re-graduate the clocks of two *approaching* observers so as to make them appear as relatively stationary, and that thereby the fact of ultimate collision would be ignored. But it no more means that the fact of ultimate collision is denied than in the paradox of Achilles and the tortoise the ultimate overtaking is denied. It simply means that the language to which this particular regraduation would lead would be unsuitable for describing the ultimate collision. Actually the possi-bility of collision of two equivalent particles is fully taken into account in the theory of regraduation. For regraduation is actually applied only to the members of what is technically called 'an equiva-lence', and it is shown in the theory of equivalences that if ever two members of an equivalence coincide, then at the same instant all coincide, and the phenomenon would be called the 'end of the world'. But the substratum in t-measure is essentially an expanding system, and no such paradox is encountered by the regraduation from t to τ. So long as we restrict ourselves to dynamical phenomena, there appears to be an unending reservoir of past time available, but in phenomena customarily described on the t-scale only an amount of time up to a certain value of t will be available.

255. The Lorentz formulae. The reason for deriving the Lorentz formulae in terms of clock-measures only has also been much mis-understood. It would have been possible to construct the same theory, so far as it is confined to t-measure, by plunging *in medias res* and assuming the Lorentz formulae outright. But then we should have arrived at descriptions of phenomena which would not have coincided with the descriptions customary in physics—for example in the description of the acceleration of a free particle. It would then have been necessary to reinvestigate the basis of the Lorentz formulae. We should thus have been compelled to go back in the end to our present starting-point, and we should have found that the Lorentz formulae could be derived on a time basis only.

256. Certain logical possibilities excluded. The present theory has also been criticized as excluding the possibility of the universe

occupying a closed spherical space. The *logical* possibility of using a closed space of positive curvature is not denied, any more than the logical possibility of describing a four-dimensional cube. But we are not concerned in this book with the whole arena of logical possibilities. It is *logically possible* that when a light-signal is dispatched and intercepted by a distant particle, the reflected signal may return at two or more different times, instead of there being a unique reflected signal. But it would be a needless searching for complications to introduce such a possibility at the outset. Once we have embarked on our investigation of the Lorentz formulae, we find no turning-points where a bifurcation of progress is possible, just as if we begin with the simple assumption that space is three-dimensional, we do not encounter a four-dimensional cube. It is always possible for an observer to adopt for the description of his own observations a private Euclidean space, and when he and all equivalent observers do so, they find, as the most convenient *public* space for the description of the substratum, a hyperbolic space in τ-measure, of curvature $(cl_0)^{-2}$.

257. Philosophy of perception put into practice. The derivation of the Lorentz formulae on a time-basis involves just those perceptions of elementary sense-data—elementary acts of seeing—which the philosophers are always talking about. The derivation amounts to philosophy put into practice. Moreover, Lorentz-invariance and the demand for it plays so large a part in physics that one would have expected the application of it to the grandest of all phenomena, the phenomenon of the expanding universe itself, to be welcomed by physicists as well as philosophers. It has indeed been criticized mostly by those who appear to have a vested interest in the cumbrous machinery of 'general' relativity. That the present methods are simpler than those of 'general' relativity is tacitly admitted by those who insist that the present theory is a particular case of 'general' relativity. But the universe *is* a particular case; there is only one universe; and it has certainly been worth while to explore the full consequences of the model called the substratum before losing ourselves in the endless sands of 'general' relativity.

258. The inverse square law of gravitation. The particularization of the function $G(\xi)$ in the acceleration formula to the value $G(\xi) \equiv -1$, corresponding to a pure substratum, was accompanied

with the expression of the inverse square law of gravitation in Lorentz invariant form, and the evaluation of the Newtonian constant of gravitation γ_0 in terms of c, t_0 and the apparent mass of the universe, M_0, in the form $\gamma_0 = c^3 t_0 / M_0$. This is a constant on the τ-scale, but its value depends on the epoch at which the transition to the τ-scale from the t-scale is made. On the t-scale, γ_0 is replaced by a variable $\gamma = c^3 t / M_0$, so that fundamentally the so-called 'constant' of gravitation varies proportionally to the epoch. The occurrence of M_0 in the formula for γ realizes Mach's expectation that gravitation is a consequence of the total amount of matter in the universe. The formula is best used numerically by letting it determine a value for M_0; taking $\gamma_0 = 6 \cdot 6 \times 10^{-8}$ cm.^{3} sec.$^{-2}$ gram^{-1}, $c = 3 \times 10^{10}$ cm. sec., $t_0 = 2 \times 10^9$ years $= 6 \cdot 3 \times 10^{10}$ sec., we have

$$M_0 = \frac{c^3 t_0}{\gamma_0} = 2 \cdot 6 \times 10^{55} \text{ grams},$$

which is comparable with the value assigned to the mass of the whole universe on the theory of the Einstein spherical universe. It must be remembered, however, that in the present theory the mass of the whole universe is infinite; M_0 is the *apparent* mass obtained by filling the apparent volume with a density equal to the density near the observer, this mass being a constant independent of the epoch at which the observer evaluates the density. (At a later epoch, for example, though the density will be smaller, the volume will be larger in the same proportion.) The above value of M_0 corresponds to a present mean value of the density near ourselves of 10^{-27} gram cm.$^{-3}$

259. Limitation of present theory of gravitation.

The analysis which led to the connexion between $G(\xi)$ and $\psi(\xi)$ itself threw up a form of the inverse square law of gravitation in Lorentz-invariant form for transformations from any one fundamental observer to any other. The possibility of this invariance of a gravitational potential is intimately connected with the occurrence of a 'constant' of gravitation γ proportional to the epoch t on the t-scale. Lorentz-invariance of the gravitational potential under transformations from one fundamental observer to another is thus a property holding good in t-measure; this accounts for previous failures to express the inverse square law in Lorentz-invariant form, for such an expression is not possible until the two scales of time are isolated. The invariant

expression for the attraction between two particles reduces to an exact inverse square law when the observer is chosen to be at the attracting particle. This suggests that the present formulation of the theory of *local* gravitation applies only when the attracting particle is a fundamental particle, i.e. a particle at local cosmical rest. It should therefore give the orbits round a nebular nucleus, and throw light on the spiral structure, but it will not necessarily give in all detail the orbits round a non-fundamental particle, say the orbits of planets round a sun which is not at local cosmical rest.

260. Energy as an invariant. A feature of the t-dynamics is that in it *energy* is an invariant, taking the same numerical value for all fundamental observers; energy is *not* the fourth or time-component of a 4-vector. Moreover, the energy of a free particle remains constant along its trajectory, in the t-dynamics. This energy reduces to Einstein's expression for energy for a particle passing close to a fundamental particle, whether the observer is at that fundamental particle or not. Thus the swiftly receding distant nebular nuclei are not to be thought of as reservoirs of a large amount of kinetic energy; every fundamental particle or idealized nebular nucleus has the same energy. Einstein's relation between mass and energy holds good exactly in the t-dynamics. There is thus place for a world-wide theorem of conservation of energy, and of mass. This is not so in Einstein's special relativity mechanics, where the energy depends on the velocity and so varies with the frame of reference employed.

261. Non-reversibility of motion in t-dynamics. The equation of motion of a free particle in the t-dynamics is *not* unaltered in form when the sign of dt is changed. The path of a particle determined on this dynamics is thus not reversible. This is obvious physically; for the motion of a particle will only be reversed if the motion of *every* particle in the universe is reversed at the same time. This would mean reversing the expansion into a contraction. In the τ-dynamics, on the other hand, the fundamental particles are relatively stationary, and so reversing the motion of a free particle requires no further reversals of motion to give a retraced trajectory. This accounts at once for the reversibility of the motion of a particle on Newtonian mechanics. It is an additional reason for identifying the time-variable in Newtonian mechanics with the time-variable (τ) in which the substratum is relatively stationary.

262. Absolute simultaneity. In the relatively stationary substratum there is an absolute simultaneity, so that events with the same value of τ to any one fundamental observer have the same value of τ for any other fundamental observer. This is an immediate consequence of the absence of relative motion. But it must be remembered that the frames of reference for which this is true are the frames associated with the nebular nuclei, or the points of local cosmical rest. It is thus possible to pick out at each locality a frame such that in the universe of these frames Newtonian time holds sway. We solve the question of the existence of 'inertial frames' and of the existence of Newtonian time by one and the same procedure, by the selection of frames at local cosmical rest.

263. Spiral character of the galaxies. Though the theory of gravitation we have been led to does not deal in a refined way with the details of planetary orbits, as at present developed, it does open the way to an understanding of the spiral nature of the galaxies. By investigating the consequences of E. W. Brown's hypothesis that spiral arms are envelopes of orbits, we were led to the view that spiral arms must be the present positions of particles (stars) whose orbits *cannot* have had an envelope. Using the kinematic theory of gravitation, we found that the only circumstances in which an envelope was avoided were those in which all the particles had been emitted from a fixed point (r_1, β) at various epochs t_1. We found that whilst the *orbit* of a particle emitted from (r, β) at epoch t_1 took the form

$$\theta = \beta + \left(\frac{\gamma_1 M t_1^2}{r_1^3}\right)^{\frac{1}{2}} \log\frac{t}{t_1}, \qquad r = r_1\frac{t}{t_1},$$

γ_1 being the value of the constant of gravitation at epoch t_1 and M the nuclear mass, the *spiral arm* at any epoch t took the form

$$\theta = \beta + \left(\frac{\gamma_t M t^2}{r^3}\right)^{\frac{1}{2}} \log\frac{r}{r_1}.$$

The *present* position (θ_0, r_0) of the particle emitted at time t_1 is accordingly

$$\theta_0 = \beta + \left(\frac{\gamma_1 M t_1^2}{r_1^3}\right)^{\frac{1}{2}} \log\frac{t_0}{t_1}, \qquad r_0 = r_1\frac{t_0}{t_1},$$

and the present position of the spiral arm is

$$\theta_0 = \beta + \left(\frac{\gamma_0 M t_0^2}{r_0^3}\right)^{\frac{1}{2}} \log\frac{r_0}{r_1},$$

the two being reconcilable in virtue of the relations

$$\gamma_1 = \frac{c^3 t_1}{M_0}, \qquad \gamma_0 = \frac{c^3 t_0}{M_0}, \qquad \frac{r_0}{r_1} = \frac{t_0}{t_1}.$$

It will be seen that the secular variation of γ with t plays an essential part in this description of the relation of the orbits to the spiral arm.

264. Reversal of sense of winding of a spiral arm. The spiral arm has a number ν of convolutions in the orbital direction, given at epoch t by

$$\nu = \frac{1}{3\pi e}\left(\frac{\gamma_t M t^2}{r_1^3}\right)^{\frac{1}{2}},$$

followed by an equal number of convolutions in the opposite sense. This number of convolutions may be written in the form

$$\nu = \frac{1}{3\pi e}\left(\frac{\text{density of nucleus at time } t_1}{\text{mean density of universe at time } t,\text{ near ourselves}}\right)^{\frac{1}{2}},$$

and gives ν of the observed order of magnitude, in general. The turning-point in the spiral arm is reached the more suddenly the larger is ν, and matter outside the turning-point should have been emitted at times t_1 earlier than $e^{-\frac{1}{2}}t_0 = 0\cdot5134t_0$, matter belonging to the portion of an arm inside the turning-point should have been emitted at times t_1 later than $0\cdot5134t_0$.

Observational evidence has been independently brought forward by Lindblad that some spiral nebulae do in fact show such a reversal of the sense of winding of the spiral arms. He has found that certain long-exposure photographs of certain spirals show a sense of winding of the outer faint arms opposite in sense to the brighter inner arms shown up in short-exposure photographs. It is not necessary, however, for every nebula to exhibit both sets of convolutions; this depends on the evolutionary history of the nebula.

265. Significance of t-measure. The equations to the orbits and spiral arm given above are expressed in t-measure. This of course is the more fundamental measure, as it does not involve mention of the constant t_0. It is worthy of notice that the equation to the spiral arm at time t contains no mention of t_0. It represents therefore a *permanent* equation for a nebular arm. It is most satisfactory that not only does the expansion of the universe, the recession of the nebulae, receive its simplest explanation in t-measure, but also the explanation of the spiral arms requires t-measure. Nature, as it were, knows only t-measure in these cosmological questions.

266. The photon. A further cosmological phenomenon receives its explanation in the present monograph—namely Hubble's apparent paradox that in spite of the red-shift, the description of the universe as not expanding is more acceptable than the description of it as expanding. The explanation is that Hubble has mistakenly corrected his observed luminosities for what he calls the 'energy-effect', as well as for the 'motion-effect' properly due to the recession. On the formulation of dynamics in the present volume, the *energy* of a free particle, in particular of a light-particle or photon, remains constant in its transit through any portion of the universe. Although the photon suffers a red-shift, it suffers no loss of energy; this is reconcilable with the quantum relation $E = hn$ only on the view that h changes secularly with t, a result which is also required on the present dynamics by the fact that angular momentum increases secularly with t, and h is a measure of angular momentum. Thus luminosities of nebulae require correction by only one factor, the 'number-factor' due to the recession. Hubble found that when he used only one correcting factor he had a much more likely picture of the distribution of the galaxies in space than when he used two; when he used two factors he obtained an improbably small universe with an improbably short time-scale. The *single* factor that he used, which he mistakenly called the 'energy-factor', was really the numerically equal 'recession-factor', which is required fundamentally if the universe is expanding in t-measure; in t-measure, no energy-factor should be required.

Thus we have connected in one body of theory the observed recession, the observed spiral form, and the observed distribution of nebulae in depth. It has been emphasized by Vogt that these cosmological problems ought to be interconnected. We did not set out to *seek* explanations of observed cosmological phenomena. They have arisen in an unforced way, as a consequence of laying a firm kinematical foundation.

267. Cosmic rays. In *World-Structure* (the volume to which the present volume is a sequel) I attributed the corpuscular component of cosmic rays to particles, either neutral or charged, which had been accelerated up to nearly the speed of light by the gravitational fields existing in the vast inter-nebular spaces. I see no reason to alter this view. But there remains the undulatory component of cosmic

rays. Blackett has pointed out, in verbal discussion of the problem of the origin of this undulatory component, that on existing physics a serious difficulty is offered by the red-shift itself: if the undulatory components originate in some process occurring in very distant galaxies, the photons concerned should be subject to a large red-shift, which would, according to contemporary physics, reduce their energies by large factors, and leave the large energies with which they appear to arrive quite unexplained. What is needed, to explain the energy of the undulatory component, Blackett has stated, is a 'blue-shift'! That is, one needs an effect to counteract the decrease of energy supposed to be associated with the red-shift. Now the present studies have not isolated any 'blue-shift'; but they have isolated an effect which counteracts the supposed loss of energy associated with the red-shift. For, according to the results of Chapter VIII above, the photon always conserves its energy, in spite of the red-shift: the Doppler effects do not imply any corresponding reduction of energy. Consequently, photons emitted when the world was young, which would in consequence have small wave-lengths, large frequencies, and so large energies, would retain those energies in their transit through the universe before being absorbed by the apparatus of some cosmic-ray observer. The present theory thus goes half-way towards removing Blackett's difficulty: it destroys the consequences in energetics of the all-pervading red-shift. The idea clearly needs much more thought, but it would not be permissible here to pass it over in silence.

268. General aspects. But over and above these particular facts concerning galaxies, the present theory gives an insight into general physical and dynamical relationships in the universe at large. It explains why the laws of dynamics come to be what they are. It explains why the laws of gravitation come to be what they are. It distinguishes between ephemeral relationships, discovered and expressed in terms of the ephemeral (τ, t_0) scale of time, and more fundamental relationships expressed in the t-scale of time. These latter relationships, containing as they do mention of the epoch t to which they refer, bear witness to the creation of the world in time, taking due account of the actual origin of things. The world is thus no static mechanism, though certain phenomena can be presented under such a dress; it is an evolving mechanism, dependent on its

own age, and containing, as I think it does, countless examples of its fundamental constituents, the galaxies, it provides infinite opportunities for the play of evolution. As they appear to us, the overwhelming majority of these galaxies have only just been created; they are, in our present, early in time. *Their* creation is only just a thing of the past. For them the drama of evolution is only just begun. They present the spectacle of unnumbered experiments in evolution; and though each is destined to age, and possibly to come under the sway of the second law of thermodynamics, there is no sense in which the second law of thermodynamics applies to the universe as a whole.

The present essay is expressed in terms of reverent optimism. It has nowhere mentioned God. The First Cause of the universe is left for the reader to insert. But our picture is incomplete without Him. It requires a more powerful God to create an infinite universe than a finite universe; it requires a greater God to leave room for an infinity of opportunities for the play of evolution than to wind up a mechanism, once and for all. We rescue the idea of God from the littlenesses that a pessimistic science has in the past placed upon Him.

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